Operations Research I: Models & Applications
Linear Programming

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Introduction

- Let's study **Linear Programming** (LP).
  - It is used a lot in practice.
  - It also possesses useful mathematical properties.
  - It is a good starting point for all OR subjects.
- We will study:
  - What kind of practical problems may be solved by LP.
  - How to formulate a problem as an LP.

Road map

- **Terminology**.
- The graphical approach.
- Three types of LPs.
- Simple LP formulations.
- Compact LP formulations.

Linear Programs

- Linear Programming is the process of formulating and solving **linear programs** (also abbreviated as LPs).
- An LP is a **mathematical program** with some special properties.
- Let's first introduce some concepts of mathematical programs.

Basic elements of a program

- In general, any mathematical program may be expressed as

$$\begin{array}{rlll} \min & f(x_1, x_2, ..., x_n) & & (\textbf{objective function}) \\ \text{s.t.} & g_i(x_1, x_2, ..., x_n) \leq b_i & \forall i = 1, ..., m & (\textbf{constraints}) \\ & x_j \in \mathbb{R} & \forall j = 1, ..., n. & (\textbf{decision variable}) \end{array}$$

  - There are m constraints and n variables.
  - x_1, x_2, ..., and x_n are real-valued decision variables.
  - We may write

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (x_1, ..., x_n)$$

    as a **vector** of decision variables (or a decision vector).
  - $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are all real-valued functions.
  - Mostly we will omit $x_j \in \mathbb{R}$.

Transformation

- How about a maximization objective function?
  - $\max f(x) \Leftrightarrow \min -f(x)$.
- How about "=" or "≥" constraints?
  - $g_i(x) \geq b_i \Leftrightarrow -g_i(x) \leq -b_i$.
  - $g_i(x) = b_i \Leftrightarrow g_i(x) \leq b_i$ and $g_i(x) \geq b_i$, i.e., $-g_i(x) \leq -b_i$.
- For example:

$$
\begin{array}{rrcrcr}
\max & x_1 & - & x_2 & & \\
\text{s.t.} & -2x_1 & + & x_2 & \geq & -3 \\
 & x_1 & + & 4x_2 & = & 5.
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{rrcrcr}
\min & -x_1 & + & x_2 & & \\
\text{s.t.} & 2x_1 & - & x_2 & \leq & 3 \\
 & x_1 & + & 4x_2 & \leq & 5 \\
 & -x_1 & - & 4x_2 & \leq & -5.
\end{array}
$$

Sign constraints

- For some reasons that will be clear in the next week, we distinguish between two kinds of constraints:
  - **Sign constraints**: $x_i \geq 0$ or $x_i \leq 0$.
  - **Functional constraints**: all others.
- For a variable x_i:
  - It is **nonnegative** if $x_i \geq 0$.
  - It is **nonpositive** if $x_i \leq 0$.
  - It is **unrestricted in sign** (urs.) or **free** if it has no sign constraint.

Feasible solutions

- For a mathematical program:
  - A **feasible solution** satisfies all the constraints.
  - An **infeasible solution** violates at least one constraint.
- For example:

$$
\begin{array}{rrcrcr}
\min & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 12 \\
& x_1 & - & 2x_2 & \geq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0.
\end{array}
$$

- Feasible?
  - $x^1 = (2, 3)$.
  - $x^2 = (6, 0)$.
  - $x^3 = (6, 6)$.

Feasible region and optimal solutions

- The **feasible region** (or **feasible set**) is the set of feasible solutions.
  - The feasible region may be empty.
- An **optimal solution** is a feasible solution that:
  - Attains the largest objective value for a maximization problem.
  - Attains the smallest objective value for a minimization problem.
  - In short, no feasible solution is better than it.
- An optimal solution may not be unique.
  - There may be **multiple** optimal solutions.
  - There may be **no** optimal solution.

Binding constraints

- At a solution, a constraint may be **binding**:[1]

Definition 1

Let $g(\cdot) \leq b$ be an inequality constraint and $\bar{x}$ be a solution. $g(\cdot) \leq b$ is binding at $\bar{x}$ if $g(\bar{x}) = b$.

  - An inequality is **nonbinding** at a point if it is strict at that point.
  - An equality constraint is always binding at any feasible solution.
- Some examples:
  - $x_1 + x_2 \leq 10$ is binding at $(x_1, x_2) = (2, 8)$.
  - $2x_1 + x_2 \geq 6$ is nonbinding at $(x_1, x_2) = (2, 8)$.
  - $x_1 + 3x_2 = 9$ is binding at $(x_1, x_2) = (6, 1)$.

[1] Binding/nonbinding constraints are also called **active**/inactive constraints.

Strict constraints?

- An inequality may be **strict** or **weak**:
  - It is strict if the two sides cannot be equal. E.g., $x_1 + x_2 > 5$.
  - It is weak if the two sides may be equal. E.g., $x_1 + x_2 \geq 5$.
- A "practical" mathematical program's inequalities are **all weak**.
  - With strict inequalities, an optimal solution may not be attainable!
  - What is an optimal solution of

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & x > 0? \end{aligned}$$

- Think about budget constraints.
  - You want to spend \$500 to buy several things.
  - Typically, you cannot spend more than \$500.
  - But you may spend exactly \$500.

Linear Programs

- A mathematical program

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \le b_i \quad \forall i = 1, ..., m, \end{aligned}$$

is an LP if f and g_is are all **linear** functions.

  - Each of these linear functions may be expressed as

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{j=1}^{n} a_j x_j,$$

where $a_j \in \mathbb{R}$, $j = 1, ..., n$, are the **coefficients**.
  - We may write $a = (a_1, ..., a_n)$ and $f(x) = a^T x$.

- An example:

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 6 \\ & 2x_1 + x_2 \le 6 \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

Linear Programs

- In general, an LP may always be expressed as

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} A_{ij} x_j \leq b_i \quad \forall i = 1, ..., m. \end{aligned}$$

  - A_{ij}s: **constraint coefficients**.
  - b_is: right-hand-side values (**RHS**).
  - c_js: **objective coefficients**.

- Or by **vectors**:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leq b_i \quad \forall i = 1, ..., m. \end{aligned}$$

  - $a_i \in \mathbb{R}^n$, $b_i \in \mathbb{R}$, $c \in \mathbb{R}^n$.
  - $x \in \mathbb{R}^n$.

- Or by **matrices**:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b. \end{aligned}$$

  - $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

Road map

- Terminology.
- **The graphical approach**.
- Three types of LPs.
- Simple LP formulations.
- Compact LP formulations.

Graphical approach

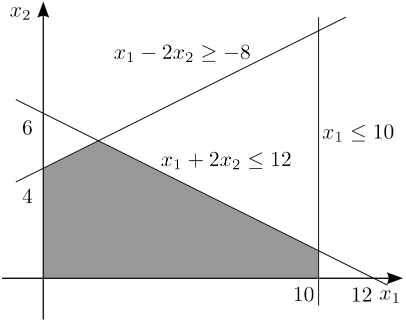
- For LPs with only two decision variables, we may solve them with the **graphical approach**.
- Consider the following example:

$$
\begin{array}{lrcrcl}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 12 \\
 & x_1 & - & 2x_2 & \geq & -8 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0.
\end{array}
$$

Graphical approach

- Step 1: Draw the feasible region.
  - Draw each constraint one by one, and then find the intersection.

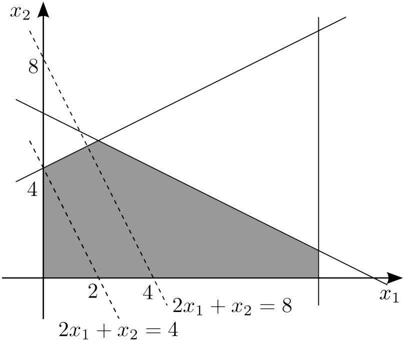
$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 12 \\
& x_1 & - & 2x_2 & \geq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0.
\end{array}
$$



Graphical approach

- Step 2: Draw some **isoquant lines**.
  - A line such that all points on it result in **the same** objective value.
  - Also called **isoprofit** or **isocost** lines when it is appropriate.
  - Also called **indifference lines** (curves) in Economics.

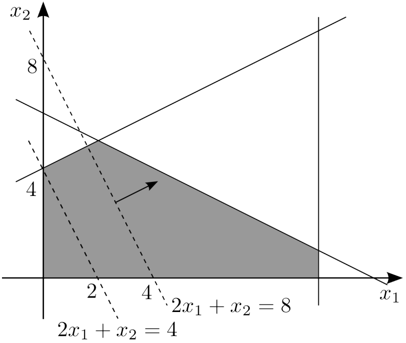
$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 12 \\
& x_1 & - & 2x_2 & \geq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0.
\end{array}
$$



Graphical approach

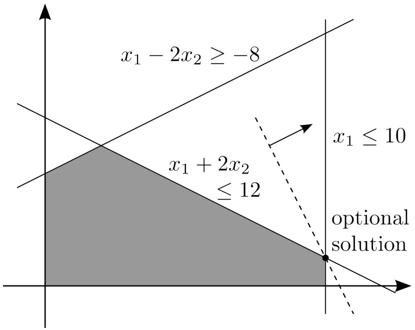
- Step 3: Indicate the direction to push the isoquant line.
  - The direction that **decreases**/increases the objective value for a **minimization**/maximization problem.

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 12 \\
& x_1 & - & 2x_2 & \geq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0.
\end{array}
$$



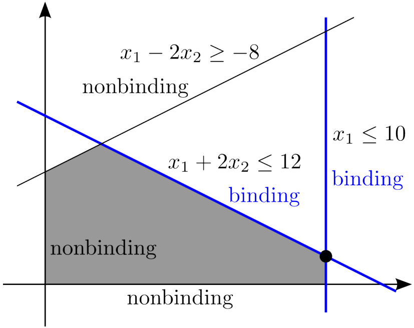
Graphical approach

- Step 4: Push the isoquant line to the "end" of the feasible region.
  - Stop when any further step makes all points on the isoquant line infeasible.



Graphical approach

- Step 5: Identify the binding constraints at an optimal solution.



Graphical approach

- Step 6: Set the binding constraints to equalities and then solve the linear system for an optimal solution.
  - In the example, the binding constraints are $x_1 \leq 10$ and $x_1 + 2x_2 \leq 12$.
  - We may solve the linear system

$$\begin{array}{rcrcl} x_1 & & & = & 10 \\ x_1 & + & 2x_2 & = & 12 \end{array}$$

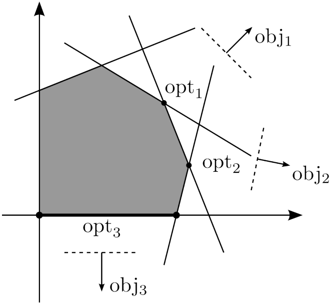
    in any way and obtain an optimal solution $(x_1^*, x_2^*) = (10, 1)$.
  - For example, through Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 1 & 2 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 1 \end{array}\right]$$

- Step 7: Plug in an optimal solution obtained into the objective function to get the associated objective value.
  - In the example, $2x_1^* + x_2^* = 21$.

Where to stop pushing?

- Where we push the isoquant line, where will be stop at?
- Intuitively, we **always** stop at a "**corner**" (or an edge).



- Is this intuition still true for LPs with more than two variables? Yes!
  - A more rigorous definition of "corners" exists.

Road map

- Terminology.
- The graphical approach.
- **Three types of LPs**.
- Simple LP formulations.
- Compact LP formulations.

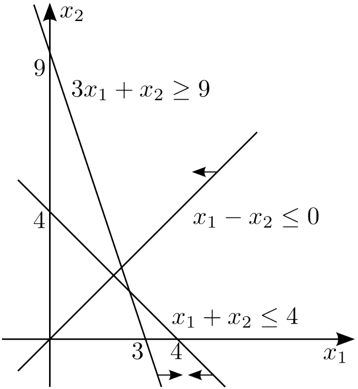
Three types of LPs

- For any LPs, it must be one of the following:
  - Infeasible.
  - Unbounded.
  - Finitely optimal (having an optimal solution).
- A finitely optimal LP may have:
  - A unique optimal solution.
  - Multiple optimal solutions.

Infeasibility

- An LP is **infeasible** if its feasible region is empty.

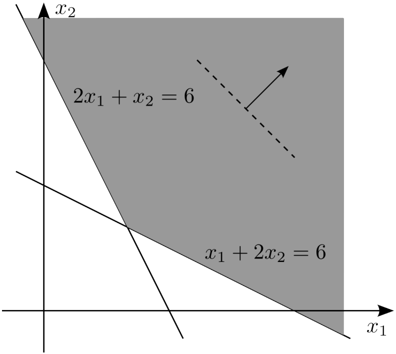
$$\begin{array}{rrcrcr} \min & 3x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 4 \\ & 3x_1 & + & x_2 & \geq & 9 \\ & x_1 & - & x_2 & \leq & 0. \end{array}$$



Unboundedness

- An LP is **unbounded** if for any feasible solution, there is another feasible solution that is better.

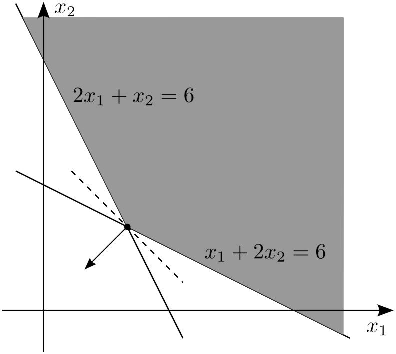
$$
\begin{array}{rrcrcl}
\max & x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\
 & 2x_1 & + & x_2 & \geq & 6.
\end{array}
$$



Unboundedness

- Note that an unbounded feasible region **does not imply** an unbounded LP!
  - Is it necessary?

$$\begin{array}{rrcrcl} \min & x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\ & 2x_1 & + & x_2 & \geq & 6. \end{array}$$

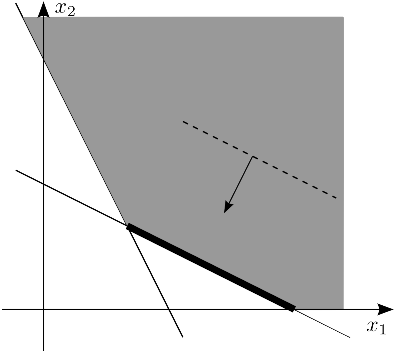


- If an LP is neither infeasible nor unbounded, it is **finitely optimal**.

Multiple optimal solutions

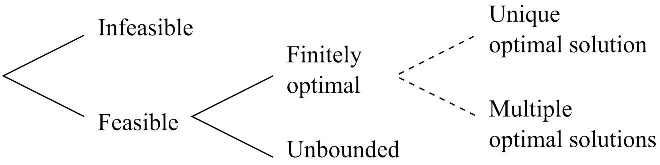
- A linear program may have **multiple** optimal solutions.

$$
\begin{array}{rrcrcl}
\min & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\
 & 2x_1 & + & x_2 & \geq & 6 \\
 & & & x_2 & \geq & 0.
\end{array}
$$



- If the slope of the isoquant line is identical to that of one constraint, will we always have multiple optimal solutions?

Summary



- In solving an LP (or any mathematical program) in practice, we only want to find **an** optimal solution, not all.
  - All we want is to make an optimal decision.

Road map

- Terminology.
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- **Simple LP formulations**.
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Introduction

- It is important to learn how to model a practical situation as an LP.
  - Once you do so, you have "**solved**" the problem.
- This process is typically called **LP formulation** or **modeling**.
- Here we will give you some examples of LP formulation.
  - Practice makes perfect!
- Then we formulate large-scale problems with **compact formulations**.

A product mix problem

- We produce several products to sell.
- Each product requires some resources. **Resources are limited**.
- We want to maximize the total sales revenue with available resources.

Problem description

- We produce desks and tables.
  - Producing a desk requires three units of wood, one hour of labor, and 50 minutes of machine time.
  - Producing a table requires five units of wood, two hours of labor, and 20 minutes of machine time.
- We may sell everything we produce.
- For each day, we have
  - Two hundred workers that each works for eight hours.
  - Fifty machines that each runs for sixteen hours.
  - A supply of 3600 units of wood.
- Desks and tables are sold at $700 and $900 per unit, respectively.

Define variables

- What do we need to decide?
- Let

$$x_1 = \text{number of desks produced in a day and}$$
$$x_2 = \text{number of tables produced in a day.}$$

- With these variables, we now try to **express** how much we will earn and how many resources we will consume.

Formulate the objective function

- We want to maximize the total sales revenue.
- Given our variables x_1 and x_2, the sales revenue is $700x_1 + 900x_2$.
- The objective function is thus

$$\max \; 700x_1 + 900x_2.$$

Formulate constraints

- For each **restriction** or **limitation**, we write a constraint:

Resource	Consumption per		Total supply
	Desk	Table	
Wood	3 units	5 units	3600 units
Labor hour	1 hour	2 hours	200 workers × 8 hr/worker = 1600 hours
Machine time	50 minutes	20 minutes	50 machines × 16 hr/machine = 800 hours

- The supply of wood is limited: $3x_1 + 5x_2 \leq 3600$.
- The number of labor hours is limited: $x_1 + 2x_2 \leq 1600$.
- The amount of machine time is limited: $50x_1 + 20x_2 \leq 48000$.
  - Use the same **unit of measurement**!

Complete formulation

- Collectively, our formulation is

$$
\begin{array}{rrcrcll}
\max & 700x_1 & + & 900x_2 & & & \\
\text{s.t.} & 3x_1 & + & 5x_2 & \leq & 3600 & \text{(wood)} \\
 & x_1 & + & 2x_2 & \leq & 1600 & \text{(labor)} \\
 & 50x_1 & + & 20x_2 & \leq & 48000 & \text{(machine)} \\
 & x_1 & & & \geq & 0 & \\
 & & & x_2 & \geq & 0. &
\end{array}
$$

- In any case:
  - **Clearly** define decision variables **in front of** your formulation.
  - Write **comments** after the objective function and constraints.

Solve and interpret

- An optimal solution of this LP is $(884.21, 189.47)$.
- So the interpretation is... to produce 884.21 desks and 189.47 tables?
- "Producing 884.21 desks and 189.47 tables" seems weird, but in fact:
  - We may produce 884.21 desks and 189.47 tables per day **in average** (i.e., roughly 88,420 desks and 18,947 tables per 100 days).
  - We may **suggest** to produce, e.g., 884 desks and 189 tables.[2]
  - It still **supports** our decision making.
  - It may not really be optimal, but we spend a very short time to make a good suggestion.
  - "All models are wrong, but some are useful."

[2] Why not 885 desks and 190 tables or the other two ways of rounding?

Produce and store!

- When we are making decisions, we may also consider what will happen in the **future**.
- This creates **multi-period** problems.
- In many cases, products produced today may be **stored** and then sold in the future.
  - Maybe daily capacity is not enough.
  - Maybe production is cheaper today.
  - Maybe the price is higher in the future.
- So the production decision must be jointly considered with the **inventory** decision.

Problem description

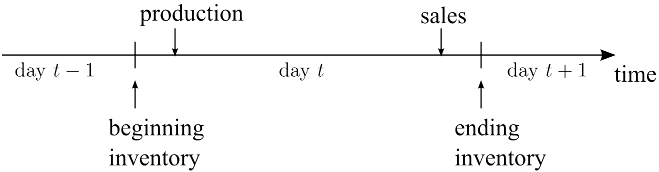
- We produce and sell a product.
- For the coming four days, the marketing manager has promised to fulfill the following amount of demands:
  - Days 1, 2, 3, and 4: 100, 150, 200, and 170 units, respectively.
- The unit production costs are different for different days:
  - Days 1, 2, 3, and 4: \$9, \$12, \$10, and \$12 per unit, respectively.
- The prices are all **fixed**. So maximizing profits is the same as minimizing costs.
- We may store a product and sell it later.
  - The **inventory cost** is \$1 per unit per day.[3]
  - E.g., producing 620 units on day 1 to fulfill all demands costs

$$\$9 \times 620 + \$1 \times 150 + \$2 \times 200 + \$3 \times 170 = \$6,640.$$

[3] Where does this inventory cost come from?

Problem description: timing

- Timing:



  - Beginning inventory + production − sales = ending inventory.
  - Inventory costs are calculated according to **ending inventory**.

Variables and objective function

- Let

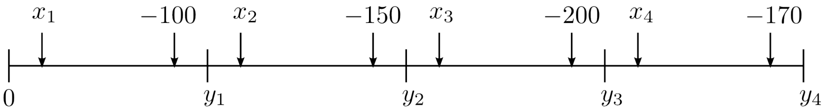
$$x_t = \text{production quantity of day } t, t = 1, ..., 4.$$
$$y_t = \textbf{ending} \text{ inventory of day } t, t = 1, ..., 4.$$

  - It is important to specify "ending"!
- The objective function is

$$\min \ 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4.$$

Constraints

- We need to keep an eye on our inventory:



  - Day 1: $x_1 - 100 = y_1$.
  - Day 2: $y_1 + x_2 - 150 = y_2$.
  - Day 3: $y_2 + x_3 - 200 = y_3$.
  - Day 4: $y_3 + x_4 - 170 = y_4$.
- These are typically called **inventory balancing** constraints.
- We also need to fulfill all demands at the moment of sales:
  - $x_1 \geq 100$, $y_1 + x_2 \geq 150$, $y_2 + x_3 \geq 200$, and $y_3 + x_4 \geq 170$.
- Also, production and inventory quantities cannot be negative.

The complete formulation

- The complete formulation is

$$
\begin{aligned}
\min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4 \\
\text{s.t.} \quad & x_1 - 100 = y_1 \\
& y_1 + x_2 - 150 = y_2 \\
& y_2 + x_3 - 200 = y_3 \\
& y_3 + x_4 - 170 = y_4 \\
& x_1 \geq 100 \\
& y_1 + x_2 \geq 150 \\
& y_2 + x_3 \geq 200 \\
& y_3 + x_4 \geq 170 \\
& x_t, y_t \geq 0 \quad \forall t = 1, ..., 4.
\end{aligned}
$$

Simplifying the formulation

- May we simplify the formulation?
- Inventory balancing and nonnegativity imply demand fulfillment!
  - E.g., in day 1, $x_1 - 100 = y_1$ and $y_1 \geq 0$ means $x_1 \geq 100$.
- So the formulation may be simplified to

$$
\begin{aligned}
\min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4 \\
\text{s.t.} \quad & x_1 - 100 = y_1 \\
& y_1 + x_2 - 150 = y_2 \\
& y_3 + x_3 - 200 = y_3 \\
& y_3 + x_4 - 170 = y_4 \\
& x_t \geq 0, y_t \geq 0 \quad \forall t = 1, ..., 4.
\end{aligned}
$$

- Identifying **redundant** constraints (removing them does not alter the feasible region) helps reduce the complexity of a program.

Simplifying the formulation

- One may further argue that there is no need to have ending inventory in period 4 (because it is costly but useless).
- So the formulation may be further simplified to

$$\begin{aligned} \min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 \\ \text{s.t.} \quad & x_1 - 100 = y_1, y_1 + x_2 - 150 = y_2 \\ & y_3 + x_3 - 200 = y_3, y_3 + x_4 - 170 = 0 \\ & x_t \geq 0 \quad \forall t = 1, ..., 4 \\ & y_t \geq 0 \quad \forall t = 1, ..., 3. \end{aligned}$$

- However, this is not always suggested (at this stage).
  - It is not required because a solver will see this.
  - It is too difficult if the instance scale is large.
- In summary, simplification is **good** but in most cases **unnecessary**.

Personnel scheduling

- We are scheduling employees in a department store.
  - Each employee must work for **five consecutive days** and then take rests for two consecutive days.
  - The number of employees required for each day:

Mon	Tue	Wen	Thu	Fri	Sat	Sun
110	80	150	30	70	160	120

- There are seven **shifts**: Monday to Friday, Tuesday to Saturday, ..., and Sunday to Thursday.
- We want to **minimize** the **number of employees hired**.

Personnel scheduling

- We may find a feasible solution easily.
  - For example, we may assign 150 employees to work from Monday to Friday and 160 to work from Saturday to Wednesday:

	Mon	Tue	Wen	Thu	Fri	Sat	Sun
Demand	110	80	150	30	70	160	120
Shift 1	150	150	150	150	150		
Shift 6	160	160	160			160	160
Total	310	310	310	150	150	160	160

- This solution is feasible but seems to be bad.

Decision variables and objective function

- Let Monday be day 1, Tuesday be day 2, etc.
- Let x_i be the number of employees who starts to work from day i for five consecutive days.
  - x_i is the number of employees assigned to shift i.
- The objective function is thus:

$$\min \; x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7.$$

Constraints

- Demand fulfillment:
  - 110 employees are needed on Monday:

$$x_1 + x_4 + x_5 + x_6 + x_7 \geq 110.$$

  - 80 employees are needed on Tuesday:

$$x_1 + x_2 + x_5 + x_6 + x_7 \geq 80.$$

  - 120 employees are needed on Sunday:

$$x_3 + x_4 + x_5 + x_6 + x_7 \geq 120.$$

- Nonnegativity constraints:

$$x_i \geq 0 \quad \forall i = 1, ..., 7.$$

Complete formulation

- The complete formulation is

$$
\begin{array}{rcccccccccccccccc}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & & \\
\text{s.t.} & x_1 & + & & & & & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 110 \\
& x_1 & + & x_2 & + & & & & & x_5 & + & x_6 & + & x_7 & \geq & 80 \\
& x_1 & + & x_2 & + & x_3 & + & & & & & x_6 & + & x_7 & \geq & 150 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & + & & & & & x_7 & \geq & 30 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & & & & & \geq & 70 \\
& & & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & & & \geq & 160 \\
& & & & & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & \geq & 120 \\
& \multicolumn{16}{l}{x_i \geq 0 \quad \forall i = 1, ..., 7.}
\end{array}
$$

Road map

- Terminology.
- The graphical approach.
- Three types of LPs.
- Simple LP formulations.
- **Compact LP formulations**.

Compact formulations

- Most problem instances in practice are of **large scales**.
  - The number of variables and constraints are huge.
- Many variables may be grouped together:
  - E.g., x_t = production quantity of day $t, t = 1, ..., 4$.
- Many constraints may be grouped together:
  - E.g., $x_t \geq 0$ for all $t = 1, ..., 4$.
- In modeling large-scale instances, we use **compact formulations** to enhance readability and efficiency.
- We use the following three instruments:
  - Indices $(i, j, k, ...)$.
  - Summation $(\sum)$.
  - For all $(\forall)$.

Compact objective function

- The production-inventory problem:
  - We have several periods. In each period, we first produce and then sell.
  - Unsold products become ending inventories.
  - We want to minimize the total cost.
- **Indices**: Because things will **repeat in each period**, it is natural to use an index for periods. Let $t \in \{1, ..., 4\}$ be the index of periods.
- For the objective function:

$$\min \ 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4,$$

  if we denote the unit production cost on day t as C_t, $t = 1, ..., 4$, we may rewrite it as

$$\min \ \sum_{t=1}^{4} (C_t x_t + y_t).$$

Compacting the constraints

- The original constraints:
  - $x_1 - 100 = y_1$, $y_1 + x_2 - 150 = y_2$, $y_2 + x_3 - 200 = y_3$, $y_3 + x_4 - 170 = y_4$.
- Let's denote the demand on day t as D_t, $t = 1, ..., 4$:
  - For $t = 2, ..., 4 : y_{t-1} + x_t - D_t = y_t$.
  - We cannot apply this to day 1 as y_0 is undefined!
- To group the four constraints into one compact constraint, we add an additional decision variable y_0:

$$y_t = \text{ending inventory of day } t, t = 0, ..., 4.$$

- Then the set of inventory balancing constraints are written as

$$y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., 4.$$

- Certainly we need to set up the initial inventory: $y_0 = 0$.

The complete compact formulation

- The compact formulation is

$$\begin{aligned} \min \quad & \sum_{t=1}^{4}(C_t x_t + y_t) \\ \text{s.t.} \quad & y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., 4 \\ & y_0 = 0 \\ & x_t, y_t \geq 0 \quad \forall t = 1, ..., 4. \end{aligned}$$

  - **Do not forget** those for-all statements! Without them, the formulation is wrong.
  - Nonnegativity constraints for multiple sets of variables may be combined to save some "≥ 0".
- One convention is to:
  - Use **lowercase** letters for variables (e.g., x_t).
  - Use **uppercase** letters for parameters (e.g., C_t).

Parameter declaration

- When creating parameter sets, we write something like

  denote C_t as the unit production cost on day $t, t = 1, ..., 4$.

  - Do not need to specify values, even though we have those values.
  - Need to specify the **range** through **indices**.
- Parameter declarations should be at the beginning of the formulation.
- Parameters and variables are **different**.
  - Variables are those to be determined. We do not know there values before we solve the model.
  - Parameters are given with known values.
  - Parameters are **exogenous** and variables are **endogenous**.

Compact formulation for product mix

- Consider the product mix problem.
  - Let n be the number of products and m be the number of resources.
  - Let j and i be the indices for products and resources, respectively.
  - We denote the unit sales price of product j as P_j, resource supply limit as R_i, and unit of resource i required for producing one unit of product j as A_{ij}, where $i = 1, ..., m$, $j = 1, ..., n$.
- Let x_i be the production quantity for product i, $i = 1, ..., n$.
- The compact formulation is

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} P_j x_j \\
\text{s.t.} \quad & \sum_{j=1}^{n} A_{ij} x_j \leq R_i \quad \forall i = 1, ..., m \\
& x_j \geq 0 \quad \forall j = 1, ..., n.
\end{aligned}
$$

Compact formulation for product mix

- Alternatively, let's define $J = \{1, ..., n\}$ as the set of products and $I = \{1, ..., m\}$ be the set of resources.
- The compact formulation is

$$\begin{aligned} \max \quad & \sum_{j \in J} P_j x_j \\ \text{s.t.} \quad & \sum_{j \in J} A_{ij} x_j \leq R_i \quad \forall i \in I \\ & x_j \geq 0 \quad \forall j \in J. \end{aligned}$$

Problems vs. instances

- A **problem** is an abstract description of a task to be completed or a question to be solved.
  - When we express everything with symbols, we have a problem.
- An **instance** is a concrete specification of a problem.
  - When we plug in concrete values into symbols, we obtain an instance.

- A compact formulation like

$$\begin{aligned} \max \quad & \sum_{j \in J} P_j x_j \\ \text{s.t.} \quad & \sum_{j \in J} A_{ij} x_j \leq R_i \quad \forall i \in I \\ & x_j \geq 0 \quad \forall j \in J \end{aligned}$$

describes a problem.

- A numeric formulation like

$$\begin{aligned} \max \quad & 700x_1 + 900x_2 \\ \text{s.t.} \quad & 3x_1 + 5x_2 \leq 3600 \\ & x_1 + 2x_2 \leq 1600 \\ & 50x_1 + 20x_2 \leq 48000 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

specifies an instance.