Information Economics, Spring 2013
Homework 1

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1 Rules

Note 1. This homework is due ***5:00 pm, September 13, 2013***. Please submit a hard copy into the instructor's mail box. As each team only needs to submit one copy, please indicate the names and student IDs of all team members on the first page.

Note 2. For this homework, each team can have ***at most three*** students. If your team has fewer than three students, you may be randomly teamed with students from other teams for the class discussions on September 16.

Note 3. To better control the class size, ***all*** the students who want to enroll in this course ***must*** do this homework. If one does not submit this homework, she/he will fail the course if she/he insists to take it.

2 Problems

1. (10 points) Consider the following problems regarding differentiation.
   (a) (2 points) Let $f(x) = 3x^4 + 4x^3 - x^2 + 6$. Find $\frac{d}{dx}f(x)|_{x=1}$.
   (b) (4 points) Let $f(x) = 3x_1^4 + 4x_1x_2^2 - x_2^2 + 6$. Find the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$.
   (c) (2 points) Let $f(x) = \ln(x^2 + 2)e^{2x}$. Find $\frac{d}{dx}f(x)$.
   (d) (2 points) Find the first-order Taylor expansion of $x^2 + 2x - 3$ at $x = 3$.
2. (10 points) Consider the following problems regarding integration.
   (a) (2 points) Let $f(x) = 2x^2 + 3x + 5$. Find $\int_0^2 f(x)dx$.
   (b) (2 points) Let $f(x) = e^{2x}$. Find $\int f(x)dx$ (you may ignore the constant).
   (c) (3 points) Let $f(x) = x_1x_2^2 + \sin x_1$. Find $\int f(x)dx_2$ (you may ignore the constant).
   (d) (3 points) Find $\frac{d}{dx}\int_0^x (t^2 + 2t - 3)dt$.
3. (10 points) Consider the following linear program

$$
\begin{array}{rlrcrcll}
z^* = & \min & 2x_1 & + & x_2 & & & \\
& \text{s.t.} & x_1 & + & 2x_2 & \geq & 4 & (1) \\
& & x_1 & + & x_2 & \geq & 2 & (2) \\
& & x_1 & & & \geq & 0. & (3)
\end{array}
$$

   (a) (4 points) Draw the feasible region.
   (b) (3 points) Find an optimal solution.
   (c) (3 points) Is there any redundant constraint? If so, find them.
4. (10 points) Suppose the interarrival time between consecutive bus arrivals X follows an exponential distribution with the rate five buses per hour.
   (a) (5 points) What is the probability that no bus arrives in 15 minutes?
   (b) (5 points) What is the expected number of bus arrivals in two hours?

5. (10 points) Prove or disprove that the intersection of two convex sets is a convex set.

6. (10 points) Let $f(\cdot)$ and $g(\cdot)$ be two convex functions defined over the same convex domain F. Prove or disprove that $h(x) \equiv f(x) + g(x)$ is a convex function over F.[1]

7. (10 points) Solve the following single-variate optimization problems.

   (a) (2 points) Find $\text{argmax}_{x\in\mathbb{R}}\{x^2 + 2x + 3 | x \in [-2, 1]\}$.

   (b) (2 points) Find $\text{argmax}_{x\in\mathbb{R}}\{x^2 + 2x + 3 | x \in [-2, 0]\}$.

   (c) (2 points) Find $\max_{x\in\mathbb{R}}\{x^2 + 2x + 3 | x \in [-2, 0]\}$.

   (d) (2 points) Find $\text{argmin}_{x\in\mathbb{R}}\{x^2 + 2x + 3 | x \in [-2, 1]\}$.

   (e) (2 points) Find $\text{argmin}_{x\in\mathbb{R}}\{x^2 + 2x + 3 | x \in [-2, 0]\}$.

8. (15 points) Consider the monopoly pricing problem we discussed in class. In this problem, we will show that the insight "the optimal price will (weakly) increase as the unit cost increase" is still true when the demand function $D(p)$ is twice-differentiable, nonincreasing, and concave.

   (a) (2 points) First of all, let's see the demand function $D(p)$ may indeed be concave. Suppose the consumer's valuation is no longer a uniform random variable. Instead, suppose the pdf of the valuation is $f(x) = 2x$ for $x \in [0, 1]$. Show that $f(\cdot)$ is indeed a pdf.

   (b) (3 points) Given a price $p \in (0, 1)$, what is the probability for a randomly chosen consumer to buy the product? What is the expected number of consumers who will buy the product?

   (c) (3 points) Now forget Parts (a) and (b) and work on a general $D(p)$. As $\pi(p) = (p - c)D(p)$, show that $\pi''(p) = 2D'(p) + D''(p)(p - c)$.

   (d) (2 points) Explain why $\pi(p)$ is concave if $D(p)$ is nonincreasing and concave.

   (e) (5 points) As $\pi(p)$ is concave, the optimal price is the price p satisfying $D(p)+D'(p)(p-c) = 0$, or equivalently,

   $$\frac{D(p)}{D'(p)} + p = c.$$

   Let $g(p) \equiv \frac{D(p)}{D'(p)} + p$, explain why once we show $g(p)$ is nondecreasing in p, we have shown that a higher c results in a (weakly) higher optimal price. Then show that $g(p)$ is indeed nondecreasing in p.

9. (15 points) Jensen's inequality describes a relationship between convex/concave functions and expectations. In this problem, we will prove Jensen's inequality:

   Proposition 1 (Jensen's inequality)**.** *Let $u : \mathbb{R} \to \mathbb{R}$ be a concave function and X be a continuous random variable. If $\mathbb{E}[X]$ exists, we have $\mathbb{E}\Big[u(X)\Big] \leq u\Big(\mathbb{E}[X]\Big)$.*

   (a) (5 points) Consider the following lemma: If a single-variate differentiable function $u(\cdot)$ is concave, we have

   $$u(x) \leq u(x_0) + u'(x_0)(x - x_0)$$

   for all x_0, $x \in \mathbb{R}$. Let $u(x) = -x^2 + 2x + 3$ and $x_0 = 0$, prove that the lemma is indeed true for all $x \in \mathbb{R}$. Then draw a figure to illustrate $u(\cdot)$, x_0, and the tangent line of $u(\cdot)$ at x_0.

   (b) (5 points) Prove the above lemma.[2]

   (c) (5 points) Given the above lemma, we have $u(x) \leq u(\mu) + u'(\mu)(x - \mu)$ for all $x \in \mathbb{R}$, where μ is defined to be $\mathbb{E}[X]$. Let $f(\cdot)$ be the pdf of X, we have $f(x) \geq 0$ and thus $f(x)u(x) \leq f(x)u(\mu) + f(x)u'(\mu)(x - \mu)$ for all $x \in \mathbb{R}$. Now do an integration to obtain our desired result.[3]

[1] Be aware that these functions may be non-differentiable. If they are all differentiable, is it easier to prove or disprove the statement?

[2] This lemma is actually more general: A differentiable function $u : \mathbb{R}^n \to \mathbb{R}$ is concave if and only if $u(y) \leq u(x) + \nabla u(x)(y - x)$ for all x, $y \in \mathbb{R}^n$. Nevertheless, you are only required to prove the simpler version.

[3] Though we do not prove it, $\int_{-\infty}^{\infty} g(x)f(x)dx = \mathbb{E}[g(X)]$ for all $g(\cdot)$. You may treat this as given and use it directly.