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Suggested Solution for Homework 1

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1. (a) $\nabla f(x) = \begin{bmatrix} 16x_1^3 + 2x_2^2 \\ 4x_1x_2 - 2x_2 \end{bmatrix}$, $\nabla^2 f(x) = \begin{bmatrix} 48x_1^4 & 4x_2 \\ 4x_2 & 4x_1 - 2 \end{bmatrix}$.
   (b) $\frac{d}{dx} f(x) = \frac{2x}{x^2+2} \cdot e^{2x} + \ln(x^2 + 2) \cdot 2e^{2x}$.
   (c) $\int f(x)dx_1 = \frac{1}{2}x_1^2x_2^2 + \frac{1}{2}e^{2x_1}$.
   (d) $\frac{d}{dx} \int_0^x (t^3 + 3t - 2)dt = x^3 + 3x - 2$.
   (e) $\mathbb{E}[X] = 3.8$, and $\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 2.16$.
   (f) Since $\int_0^2 f(x)dx = \frac{8}{3}k = 1, k = \frac{3}{8}$. And $\mathbb{E}[X] = \int_0^2 xf(x)dx = \frac{3}{2}$.
   (g) Since $\frac{d^2}{dx^2} f(x) = 6x + 4$ is greater than 0 over $[0, \infty)$, it is convex over the region.
   (h) Since $\frac{d^2}{dx^2} g(x) = 6x - 4 \geq 0$ occurs if and only if it is over the region $[\frac{2}{3}, \infty)$, it is convex over $[\frac{2}{3}, \infty)$.

2. (a) As shown in Figure 1, the area in gray is the feasible region. Obviously, it is not a convex set since there exists some points between point a and b that do not belong to the feasible region.

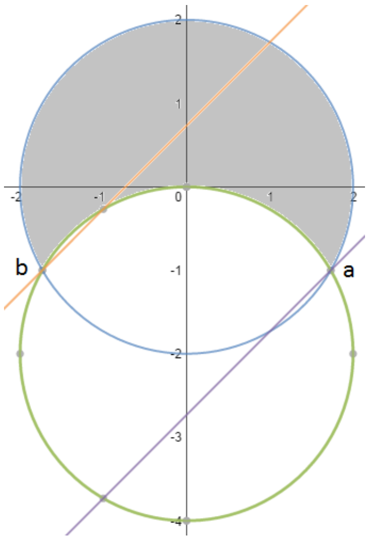

Figure 1: Graphical solution

   (b) The point a $(\sqrt{3}, -1)$ is an optimal solution.
   (c) The point b is not a global maximum but an local one since there does not exist any point nearby that is greater than it.
   (d) Since $\nabla f(x) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, there does not exist any point that satisfies the unconstrained FONC.

3. Let $z = \lambda x_1 + (1 - \lambda)x_2$ for some x_1, $x_2 \in F$, $\lambda \in [0, 1]$. Since

$$f(z) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda h(x_1) + (1 - \lambda)h(x_2)$$

and

$$g(z) \leq \lambda g(x_1) + (1 - \lambda)g(x_2) \leq \lambda h(x_1) + (1 - \lambda)h(x_2)$$

are true (by definition), we have

$$h(\lambda x_1 + (1-\lambda)x_2) = h(z) = \max\{f(z), g(z)\} \leq \lambda h(x_1) + (1-\lambda)h(x_2)$$

is obtained (which is exactly the definition of convex function). Therefore, $h(x)$ is convex over F.

4. (a) Let $f(x) = x^4 + 2x^3 + 1, x \in [-2,-1]$. Since $f(x)$ is convex over $[-2,1]$ (due to $f''(x) \leq 0$), and the FOC point occurs at $x = -\frac{3}{2} \in [-2,-1]$ due to $f'(-\frac{3}{2}) = 0$, we have

$$\underset{x\in[-2,-1]}{\text{argmin}} \{f(x)\} = \Big\{ -\frac{3}{2} \Big\}.$$

(b) Let $f(x) = x^4 + 2x^3 + 1$. Since $f(x)$ is convex over $[-2,-1]$ and strictly increasing over $[-1,0]$, the maximum point must occur at the border of the region. And because of $f(0) = f(-2) = 1$, we have

$$\underset{x\in[-2,0]}{\text{argmax}}\{f(x)\} = \{0,-2\}.$$

(c) Let $f(x) = x^4 + 2x^3 + 1, x \in [-2,1]$. Compare the points satisfying the FONC ($x = -\frac{3}{2}$ or 0) and the boundary points ($x = -2$ or 1). Since $f(-\frac{3}{2})$ is the smallest, we have

$$\underset{x\in[-2,1]}{\text{argmin}}\{f(x)\} = \Big\{ -\frac{3}{2} \Big\}.$$

5. (a) The problem can be formulated as

$$\begin{aligned} \max \quad & f(q) = (a - bq - c)q \\ \text{s.t.} \quad & a - bq \geq 0 \\ & q \geq 0. \end{aligned}$$

(b) Since $f''(q) = -2b < 0$ and $q \in [0, \frac{a}{b}]$, the problem is concave function with convex set. Therefore, it is a convex program.

(c) Since $f'(q^*) = a - 2bq - c = 0$ occurs at $q^* = \frac{a-c}{2b}$, and q^* satisfies the two constraints, the optimal production quantity is $q^* = \frac{a-c}{2b}$.

(d) q^* increases in a and decreases in b and c. When the base of the market is bigger (either out of increased a or decreased b), it will be easier for the seller to produce a larger quantity. Moreover, it is obvious that a larger production cost leads to a larger price.