IM 7011: Information Economics (Fall 2014)

Introduction and Review of Optimization

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Road map

- **Syllabus**.
- Quiz.
- Convexity.
- Optimization problems.
- Optimality conditions.

Welcome!

- This is **Information Economics**, NOT **Information Economy**.
  - We talk about IT, IS, information goods, etc.
  - We talk about **information**.
- We focus on the **economics of information**.
  - How people behave with different information?
  - What is the value of information?
  - What information to acquire? How?
  - What are the implications on the information economy?
- In this course, we focus on **information asymmetry**.

Information asymmetry

- The world is full of asymmetric information:
  - A consumer does not know a retailer's procurement cost.
  - A consumer does not know a product's quality.
  - A retailer does not know a consumer's valuation.
  - An instructor does not know how hard a student works.
- As information asymmetry results in inefficiency, we want to:
  - Analyze its impact. If possible, quantify it.
  - Decide whether it introduces driving forces for some phenomena.
  - Find a way to deal with it if it cannot be eliminated.
- This field is definitely fascinating. However:
  - We need to have some "**weapons**" to explore the world!

Before you enroll...

- Prerequisites:
  - Calculus.
  - Convex optimization.
  - Probability.
  - Game theory.
- Language: **"All" English**.
  - All materials (including course videos) are in English.
  - Students are encouraged (but not required) to speak English in class.
  - The instructor speak Chinese or English in office hour.
  - The instructor will speak Chinese in lectures when it helps.

The instructing team

- Instructor:
  - Ling-Chieh Kung.
  - Third-year assistant professor.
  - Office: Room 413, Management Building II.
  - Office hour: **10:30am-noon, Thursday** or by appointment.
  - E-mail: lckung@ntu.edu.tw.
- Teaching assistant:
  - Chia-Hao (Jack) Chen.
  - Second-year master student.
  - Office: Room 320C, Management Teaching and Research Building.
  - E-mail: r02725018@ntu.edu.tw.

Related information

- Classroom: Room 204, Management Building II.
- Lecture time: 9:10am-12:10pm, Monday.
- References:
  - *Information Rules* by C. Shapiro and H. Varian.
  - *Freakonomics* by S. Levitt and S. Dubner.
  - *Contract Theory* by P. Bolton and M. Dewatripont.
  - *Game Theory for Applied Economists* by R. Gibbons.
- Reading list:
  - Around four academic papers.
  - Around four cases.

"Flipped classroom"

- Lectures in **videos**, then discussions in classes.
- Before each Monday, the instructor uploads a video of lectures.
  - Ideally, the video will be no longer than one and a half hour.
  - Students must watch the video by themselves before that Monday.
- During the lecture, we do three things:
  - Discussing the lecture materials (0.5 to 1 hour).
  - Solving **class problems** (1 to 2 hours).
  - Further discussions (0.5 to 1 hour).
- Teams:
  - Students form teams to work on class problems and case studies.
  - Each team should have **three students**.
  - If it really helps, teams may be **reformed** by the instructor after the midterm exam.

Homework, participation, and office hour

- No homework!
  - Except Homework 1.
  - Problem sets and solutions will be posted for students to do practices.
- Class participation:
  - Just say something!
  - Use whatever way to impress the instructor.
- Office hour:
  - 10:30am-noon, Thursday.
  - Come to discuss any question (or just chat) with me!
  - If the regular time does not work for you, just send me an e-mail.
  - My "open-door" policy.

Projects and exams

- Project:
  - Please form a new team of at most n students, where the value of n will be determined according to the class size.
  - Each team will write a research proposal for a self-selected topic, make a 30-minute presentation, and submit a report.
  - All team members must be in class for the team to present.
- Two exams:
  - In-class and open whatever you have (including all electronic devices).
  - No information is allowed to be transferred among students.
  - The final exam covers only materials taught after the midterm exam.

Grading

- Homework 1: 5%.
- Class participation: 10%.
- Class problems: 20%.
- Case reports: 20%.
- Two Exams: 20% (10% each).
- Project: 25%.
- The final letter grades will be given according to the following conversion rule:

Letter	Range	Letter	Range	Letter	Range
A+	$[90, 100]$	B+	$[77, 80)$	C+	$[67, 70)$
A	$[85, 90)$	B	$[73, 77)$	C	$[63, 67)$
A−	$[80, 85)$	B−	$[70, 73)$	C−	$[60, 63)$

Important dates and tentative plan

- Important dates:
  - Week 4 (2014/10/6): No class because the instructor is in the military.
  - Week 9 (2014/11/10): Midterm exam.
  - Week 16 (2014/12/29): Final exam.
  - Weeks 17 and 18 (2015/1/5 and 2015/1/12): Project presentations.
- Tentative plan:
  - Decentralization and inefficiency.
  - The screening theory.
  - Pricing and versioning information goods.
  - The signaling theory
  - Recognizing and managing lock-in.

Related courses that are not that math-intensive

- "Electronic Commerce" by professor Ming-Hui Huang.
  - Wednesday afternoon.
  - In English.
  - No math, no paper, full of cases.
- "Revenue Management and Pricing" by professor Chia-Wei Kuo.
  - Thursday morning.
  - In Chinese.
  - Some math, no paper, full of cases.
- Also "Strategic Management," "Industrial Economics," etc.

Online resources

- CEIBA.
  - Viewing your grades.
  - Receiving announcements.
- http://www.ntu.edu.tw/~lckung/courses/IE-Fa14/.
  - Downloading course materials.
- The bulletin board "NTUIM-lckung" on PTT.
  - Discussions.
- YouTube:
  - Watching lecture videos.

Road map

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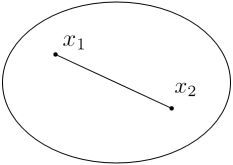
Convex sets

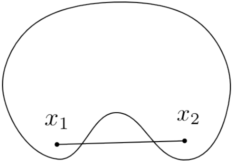
Definition 1 (Convex sets)

A set F is **convex** *if*

$$\lambda x_1 + (1-\lambda)x_2 \in F$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.





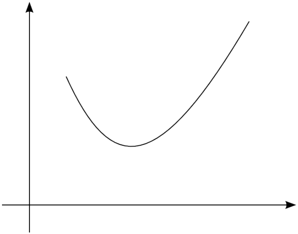
Convex functions

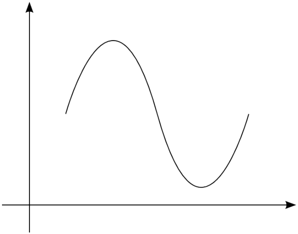
Definition 2 (Convex functions)

For a convex domain F, a function $f(\cdot)$ is **convex** *over F if*

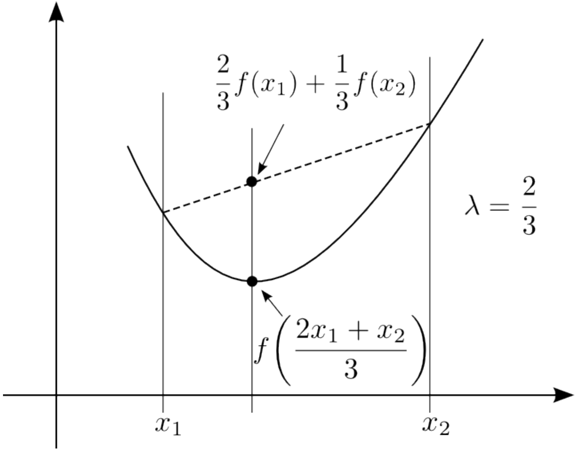
$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.





Convex functions



Some examples

- Convex sets?
  - $X_1 = [10, 20]$.
  - $X_2 = (10, 20)$.
  - $X_3 = \mathbb{N}$.
  - $X_4 = \mathbb{R}$.
  - $X_5 = \{(x, y) | x^2 + y^2 \leq 4\}$.
  - $X_6 = \{(x, y) | x^2 + y^2 \geq 4\}$.
- Convex functions?
  - $f_1(x) = x + 2, x \in \mathbb{R}$.
  - $f_2(x) = x^2 + 2, x \in \mathbb{R}$.
  - $f_3(x) = \sin(x), x \in [0, 2\pi]$.
  - $f_4(x) = \sin(x), x \in [\pi, 2\pi]$.
  - $f_5(x) = \log(x), x \in (0, \infty)$.
  - $f_6(x, y) = x^2 + y^2, (x, y) \in \mathbb{R}^2$.

Strictly convex and concave functions

Definition 3 (Strictly convex functions)

For a convex domain F, a function $f(\cdot)$ is **strictly convex** *over F if*

$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) < \lambda f(x_1) + (1-\lambda) f(x_2)$$

for all $\lambda \in (0,1)$ and $x_1, x_2 \in F$ such that $x_1 \neq x_2$.

Definition 4 ((Strictly) concave functions)

For a convex domain F, a function $f(\cdot)$ is **(strictly) concave** *over F if $-f(\cdot)$ is (strictly) convex.*

Derivatives of convex functions

- When a function is twice-differentiable, testing its convexity is simple:

Proposition 1

Consider a single-variate twice-differentiable function $f(\cdot)$ over an interval $F = [a, b]$:

- *$f(\cdot)$ is convex over F if and only if $f''(x) \geq 0$ for all $x \in F$.*
- *$f(\cdot)$ is strictly convex over F if and only if $f''(x) > 0$ for all $x \in F$.*

Proposition 2

Consider a single-variate twice-differentiable function $f(\cdot)$ over an interval $F = [a, b]$:

- *$f(\cdot)$ is concave over F if and only if $f''(x) \leq 0$ for all $x \in F$.*
- *$f(\cdot)$ is strictly concave over F if and only if $f''(x) < 0$ for all $x \in F$.*

Some examples revisited

- Convex functions?
  - $f_1(x) = x + 2, x \in \mathbb{R}$.
  - $f_2(x) = x^2 + 2, x \in \mathbb{R}$.
  - $f_3(x) = \sin(x), x \in [0, 2\pi]$.
  - $f_4(x) = \sin(x), x \in [\pi, 2\pi]$.
  - $f_5(x) = \log(x), x \in (0, \infty)$.
  - $f_6(x, y) = x^2 + y^2, (x, y) \in \mathbb{R}^2$.

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- **Optimization problems**.
- Optimality conditions.

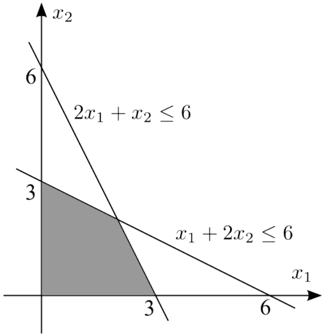
Optimization problems

- In an optimization problem, there are:
  - **Decision variables**.
  - **The objective function**.
  - **Constraints**.

Linear programming

- Consider the problem

$$\begin{aligned} z^* = \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 6 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$



  - The feasible region is the shaded area.
  - An optimal solution is $(x_1^*, x_2^*) = (2, 2)$. Is it unique?
  - The corresponding objective value $z^* = 6$.
- An optimization problem is a **linear program** (LP) if the objective function and constraints are all linear.

Nonlinear programming

- A problem is a **nonlinear program** (NLP) if it is not a linear program.
- Consider the problem

$$\begin{aligned} z^* = \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 16 \\ & x_1 + x_2 \geq 1. \end{aligned}$$

  - What is the feasible region?
  - What is an optimal solution? Is it unique?
  - What is the value of z^*?
- An optimization problem is a **convex program** if in it we maximize a concave function over a convex feasible region.
- All convex programs can be solved efficiently.
- It may not be possible to solve a nonconvex program efficiently.

Infeasible and unbounded problems

- Not all problems have an optimal solution.
- A problem is **infeasible** if there is no feasible solution.
  - E.g., $\max\{x^2 | x \leq 2, x \geq 3\}$.
- A problem is **unbounded** if given any feasible solution, there is another feasible solution that is better.
  - E.g., $\max\{e^x | x \geq 3\}$.
  - How about $\min\{\sin x | x \geq 0\}$?
- A problem may be infeasible, unbounded, or finitely optimal (i.e., having at least one optimal solution).

Set of optimal solutions

- The **set of optimal solutions** of a problem $\max\{f(x)|x \in X\}$ is

$$\operatorname{argmax}\{f(x)|x \in X\}.$$

- For $f(x) = \cos x$ and $X = [0, 2\pi]$, we have

$$\operatorname{argmax}\left\{\cos x \middle| x \in [0, 2\pi]\right\} = \{0, 2\pi\}.$$

- If x^* is an optimal solution of $\max\{f(x)|x \in X\}$, we should write

$$x^* \in \operatorname{argmax}\{f(x)|x \in X\},$$

NOT $x^* = \operatorname{argmax}\{f(x)|x \in X\}$!

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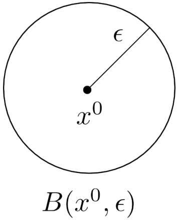
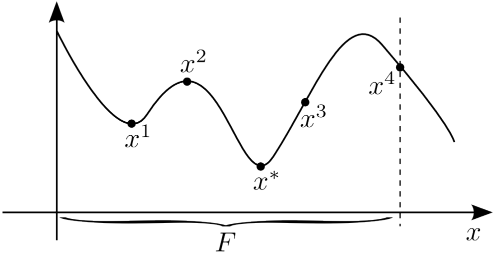
Global optima

- For a function $f(x)$ over a feasible region F:
  - A point x^* is a **global minimum** if $f(x^*) \leq f(x)$ for all $x \in F$.
  - A point x' is a **local minimum** if for some $\epsilon > 0$ we have

$$f(x') \leq f(x) \quad \forall x \in B(x', \epsilon) \cap F,$$

where $B(x^0, \epsilon) \equiv \{x | d(x, x^0) \leq \epsilon\}$ and $d(x, y) \equiv \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.





- **Global maxima** and **local maxima** are defined accordingly.

First-order necessary condition

- Consider an **unconstrained** problem

$$\max_{x \in \mathbb{R}^n} f(x).$$

Proposition 3 (Unconstrained FONC)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. For a point x^* to be a local maximum of f, we need:

- $f'(x^*) = 0$ *if* $n = 1$.
- $\nabla f(x^*) = 0$ *if* $n \geq 2$.

- For an n-dimensional differentiable function f, its **gradient** is

$$\nabla f \equiv \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}.$$

Examples

- Consider the problem

$$\max_{x \in \mathbb{R}} \; x^3 - \frac{9}{2}x^2 + 6x + 2$$

The FONC yields

$$3(x^2 - 3x + 2) = 0.$$

Solving the equation gives us 1 and 2 as two candidates of local maxima.

  - It is easy to see that $x^* = 1$ is a local maxima but $\tilde{x} = 2$ is NOT.

- Consider the problem

$$\max_{x \in \mathbb{R}^2} \; f(x) = x_1^2 - x_1x_2 + x_2^2 - 6x_2.$$

The FONC yields

$$\nabla f(x) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the linear system gives us $(2, 4)$ as the only candidate of local maxima.

  - Note that it is NOT necessarily a local maximum!

Second-order necessary condition

- Let's proceed further.

Proposition 4 (Unconstrained SONC)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice-differentiable. For a point x^* to be a local maximum of f, we need:

- $f''(x^*) \leq 0$ *if* $n = 1$.
- $y^T \nabla^2 f(x^*) y \leq 0$ *for all* $y \in \mathbb{R}^n$ *if* $n \geq 2$.

- For an n-dimensional function $f(x_1, ..., x_n) : \mathbb{R}^n \to \mathbb{R}$ that is twice-differentiable, its **Hessian** is the $n \times n$ matrix

$$\nabla^2 f \equiv \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \, \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \, \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

Second-order necessary condition

- Regarding the Hessian:
  - (Calculus) If the second-order derivatives are all continuous (which will be true for almost all functions we will see in this course), the Hessian is symmetric.
  - (Linear Algebra) A symmetric matrix A is called **negative semidefinite** if $y^T Ay \leq 0$ for all $y \in \mathbb{R}^n$.
  - Therefore, if the second-order derivatives of f all exists and are continuous, the unconstrained SONC is simply requesting the Hessian to be negative semidefinite.
- In this course, we will not apply the SONC a lot.
- Here our point is that a local maximum requires NOT just

$$\frac{\partial^2 f}{\partial x_i^2} \leq 0 \quad \forall i = 1, ..., n.$$

We want more than candidates!

- The FONC and SONC produce candidates of local maxima/minima.
- What's next?
  - We need some ways to **ensure** local optimality.
  - We need to find a **global** optimal solution.
- If the function is convex or concave, things are much easier:
  - Because for a differentiable concave/convex function, the FONC is necessary AND **sufficient** (thus called FOC in this case).

- Now points satisfying the FONC are locally optimal.
- We may prove that they are also **globally** optimal.

Remarks

- When you are asked to solve a problem:
  - First check whether the objective function is convex/concave. If so the problem typically becomes much easier.
- All the conditions for unconstrained problems apply to **interior** points of a feasible region.
- One common strategy for solving constrained problems proceeds in the following steps:
  - **Ignore** all the constraints.
  - Solve the unconstrained problem.
  - Verify that the unconstrained optimal solution satisfies all constraints.
- If the strategy fails, we seek for other ways.

Application: Monopoly pricing

- Suppose a monopolist sells a single product to consumers.
- Consumers are **heterogeneous** in their **willingness-to-pay**, or **valuation**, of this product.
- One's valuation, θ, lies on the interval $[0, b]$ uniformly.
  - He buys the product if and only if his valuation is above the price.
  - The total number of consumers is a.
  - Given a price p, in expectation how many consumers buy?
- The unit production cost is c.
- The seller chooses a unit price p to maximize her total expected profit.

Formulation

- The **endogenous** decision variable is p.
- The **exogenous** parameters are a, b, and c.
- The only constraint is $p \geq 0$.
- Let $\pi(p)$ be the profit under price p. What is $\pi(p)$?

- What is the complete problem formulation?

  - It is without loss of generality to **normalize** the population size a to 1.

Solving the problem

- Given that $\pi(p) = \frac{a}{b}(p - c)(b - p)$, let's show it is strictly concave:
  - $\pi'(p) =$
  - $\pi''(p) =$
- Great! Now let's ignore the constraint $p \geq 0$.
- Applying the FOC, what is the unconstrained optimal solution?
- Does p^* satisfy the ignored constraint? Is it globally optimal?

Managerial/economic implications

- The optimal price $p^* = \frac{b+c}{2}$ tells us something:
  - p^* is increasing in the highest possible valuation b. Why?
  - p^* is increasing in the unit cost c. Why?
  - p^* has nothing to do with the total number of consumer a. Why?
- The optimal profit $\pi^* \equiv \pi(p^*) = \frac{a(b-c)^2}{4b}$.
  - π^* is decreasing in c. Why?
  - π^* is increasing in a. Why?
  - How is π^* affected by b? Guess!
  - Let's answer it:

- It is these **qualitative** managerial/economic implications that matters.
- Never forget to verify your solutions with your **intuitions**.