IM 7011: Information Economics (Fall 2014)

Introduction to Game Theory

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Introduction

- Today we introduce **games** under **complete information**.
  - Complete information: All the information are publicly known.
  - They are **common knowledge**.
- We will introduce **static** and **dynamic** games.
  - Static games: All players act simultaneously (at the same time).
  - Dynamic games: Players act sequentially.
- We will illustrate the **inefficiency** caused by decentralization (lack of cooperation).
- We will show how to **solve** a game, i.e., to predict what players will do in **equilibrium**.

Road map

- **Prisoners' dilemma**.
- Static games: Nash equilibrium.
- Cournot competition.
- Dynamic games: Backward induction.
- Pricing in a supply chain.

Prisoners' dilemma: story

- A and B broke into a grocery store and stole some money. Before police officers caught them, they hided those money carefully without leaving any evidence. However, a monitor got their images when they broke the window.
- They were kept in two separated rooms. Each of them were offered two choices: **Denial or confession**.
  - If both of them deny the fact of stealing money, they will both get one month in prison.
  - If one of them confesses while the other one denies, the former will be set free while the latter will get nine months in prison.
  - If both confesses, they will both get six months in prison.
- They **cannot communicate** and they must make their choices **simultaneously**.
- All they want is to be in prison as short as possible.
- What will they do?

Prisoners' dilemma: matrix representation

- We may use the following matrix to formulate this "game":

Player 2

		Denial	Confession
Player 1	Denial	$-1, -1$	$-9, 0$
	Confession	$0, -9$	$-6, -6$

  - There are two **players**, each has two possible **actions**.
  - For each combination of actions, the two numbers are the **utilities** of the two players: the first for player 1 and the second for player 2.
- Prisoner 1 thinks:
  - "If he denies, I should confess."
  - "If he confesses, I should still confess."
  - "I see! I should confess anyway!"
- For prisoner 2, the situation is the same.
- The **solution** (outcome) of this game is that both will confess.

Prisoners' dilemma: discussions

- In this game, confession is said to be a **dominant strategy**.
- This outcome can be "improved" if they can **cooperate**.
- **Lack of cooperation** can result in a **lose-lose** outcome.
  - Such a situation is **socially inefficient**.
- We will see more situations similar to the prisoners' dilemma.

Solutions for a game

- Is it always possible to solve a game by finding dominant strategies?
- What are the solutions of the following games?

Player 2

Player 1		B	S
	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

Player 2

Player 1		H	T
	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

- We need a new solution concept: Nash equilibrium!

Road map

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Nash equilibrium: definition

- The most fundamental equilibrium concept is the **Nash equilibrium**:

Definition 1

For an n-player game, let S_i be player i's action space and u_i be player i's utility function, $i = 1, ..., n$. An action profile $(s_1^*, ..., s_n^*)$, $s_i^* \in S_i$, is a (pure-strategy) **Nash equilibrium** *if*

$$u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*) \\ \geq u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$$

for all $s_i \in S_i$, $i = 1, ..., n$.

- Alternatively, $s_i^* \in \operatorname*{argmax}_{s_i \in S_i} \left\{ u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*) \right\}$ for all i.
- In a Nash equilibrium, no one has an incentive to **unilaterally deviate**.
- The term "pure-strategy" will be explained later.

Nash equilibrium: an example

- Consider the following game with no dominant strategy:

Player 2

Player 1		L	C	R
	T	0, 4	4, 0	5, 3
	M	4, 0	0, 4	5, 3
	B	3, 5	3, 5	6, 6

- What is a Nash equilibrium?
  - (T, L) is not: Player 1 will deviate to M or B.
  - (T, C) is not: Player 2 will deviate to L or R.
  - (B, R) is: No one will unilaterally deviate.
  - Any other Nash equilibrium?
- Why a Nash equilibrium is an "outcome"?
  - Imagine that they takes turns to make decisions until no one wants to move. What will be the outcome?

Nash equilibrium: More examples

- Is there any Nash equilibrium of the prisoners' dilemma?

Player 2

Player 1		Denial	Confession
	Denial	$-1, -1$	$-9, 0$
	Confession	$0, -9$	$-6, -6$

- How about the following two games?

Player 2

Player 1		B	S
	B	$2, 1$	$0, 0$
	S	$0, 0$	$1, 2$

Player 2

Player 1		H	T
	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

Existence of a Nash equilibrium

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

- The last game does not have a "pure-strategy" Nash equilibrium.
- What if we allow **randomized** (mixed) strategy?

- In 1950, John Nash proved the following theorem regarding the **existence** of "mixed-strategy" Nash equilibrium:

Proposition 1

For a static game, if the number of players is finite and the action spaces are all finite, there exists at least one mixed-strategy Nash equilibrium.

  - This is a sufficient condition. Is it necessary?
- In most business applications of Game Theory, people focus only on pure-strategy Nash equilibria.

Road map

- Prisoners' dilemma.
- Static games: Nash equilibrium.
- **Cournot competition**.
- Dynamic games: Backward induction.
- Pricing in a supply chain.

Cournot Competition

- In 1838, Antoine Cournot introduced the following **quantity competition** between two retailers.
- Let q_i be the production quantity of firm i, $i = 1, 2$.
- Let $P(Q) = a - Q$ be the market-clearing price for an aggregate demand $Q = q_1 + q_2$.
- Unit production cost of both firms is $c < a$.
- Each firm wants to maximize its profit.
- Our questions are:
  - In this environment, what will these two firms do?
  - Is the outcome satisfactory?
  - What is the difference between duopoly and monopoly (i.e., decentralization and integration)?

Cournot Competition

- Players: 1 and 2.
- Action spaces: $S_i = [0, \infty)$ for $i = 1, 2$.
- Utility functions:

$$u_1(q_1, q_2) = q_1 \Big[a - (q_1 + q_2) - c\Big] \text{ and}$$
$$u_2(q_1, q_2) = q_2 \Big[a - (q_1 + q_2) - c\Big].$$

- As for an outcome, we look for a Nash equilibrium.
- If (q_1^*, q_2^*) is a Nash equilibrium, it must solve

$$q_1^* \in \underset{q_1 \in [0,\infty)}{\operatorname{argmax}} \ u_1(q_1, q_2^*) = \underset{q_1 \in [0,\infty)}{\operatorname{argmax}} \ q_1 \Big[a - (q_1 + q_2^*) - c\Big] \text{ and}$$
$$q_2^* \in \underset{q_2 \in [0,\infty)}{\operatorname{argmax}} \ u_2(q_1^*, q_2) = \underset{q_2 \in [0,\infty)}{\operatorname{argmax}} \ q_2 \Big[a - (q_1^* + q_2) - c\Big].$$

Solving the Cournot competition

- For firm 1, we first see that the objective function is strictly concave:
  - $u_1'(q_1, q_2^*) = a - q_1 - q_2^* - c - q_1$.
  - $u_1''(q_1, q_2^*) = -2 < 0$.
- The FOC condition suggests $q_1^* = \frac{1}{2}(a - q_2^* - c)$.
  - If $q_2^* < a - c$, q_1^* is optimal for firm 1.
- Similarly, $q_2^* = \frac{1}{2}(a - q_1^* - c)$ is firm 2's optimal decision if $q_1^* < a - c$.
- So if (q_1^*, q_2^*) is a Nash equilibrium such that $q_i^* < a - c$ for $i = 1, 2$, it must satisfy

$$q_1^* = \frac{1}{2}(a - q_2^* - c) \quad \text{and} \quad q_2^* = \frac{1}{2}(a - q_1^* - c).$$

- The unique solution to this system is $q_1^* = q_2^* = \frac{a-c}{3}$.
  - Does this solution make sense?
  - As $\frac{a-c}{3} < a - c$, this is indeed a Nash equilibrium. It is also unique.

Distortion due to decentralization

- What is the "cost" of decentralization?
- Suppose the two firms' are **integrated** together to jointly choose the aggregate production quantity.
- They together solve

$$\max_{Q \in [0,\infty)} \quad Q[a - Q - c],$$

  whose optimal solution is $Q^* = \frac{a-c}{2}$.
- First observation: $Q^* = \frac{a-c}{2} < \frac{2(a-c)}{3} = q_1^* + q_2^*$.
- Why does a firm intend to **increase** its production quantity under decentralization?

Inefficiency due to decentralization

- May these firms improve their profitability with integration?
- Under decentralization, firm i earns

$$\pi_i^D = \frac{(a-c)}{3}\left[a - \frac{2(a-c)}{3} - c\right] = \left(\frac{a-c}{3}\right)\left(\frac{a-c}{3}\right) = \frac{(a-c)^2}{9}.$$

- Under integration, the two firms earn

$$\pi^C = \frac{(a-c)}{2}\left[a - \frac{a-c}{2} - c\right] = \left(\frac{a-c}{2}\right)\left(\frac{a-c}{2}\right) = \frac{(a-c)^2}{4}.$$

- $\pi^C > \pi_1^D + \pi_2^D$: The integrated system is more **efficient**.
- Through appropriate profit splitting, both firm earns more.
  - Integration can result in a **win-win** solution for firms!
- However, under monopoly the aggregate quantity is lower and the price is higher. Consumers **benefits** from **firms' competition**.

The two firms' prisoners' dilemma

- Now we know the two firms should together produce $Q = \frac{a-c}{2}$.
- What if we suggest them to produce $q_1' = q_2' = \frac{a-c}{4}$?
- This maximizes the total profit but is **not** a Nash equilibrium:
  - If he chooses $q' = \frac{a-c}{4}$, I will move to

$$q'' = \frac{1}{2}(a - q' - c) = \frac{3(a-c)}{8}.$$

  - So both firms will have incentives to unilaterally deviate.
- These two firms are engaged in a prisoners' dilemma!

Road map

- Prisoners' dilemma.
- Static games: Nash equilibrium.
- Cournot competition.
- **Dynamic games: Backward induction**.
- Pricing in a supply chain.

Dynamic games

- Recall the game "BoS":

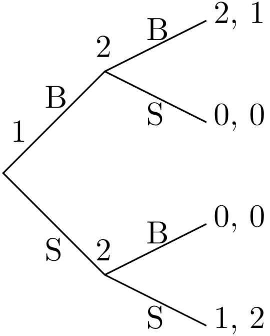
Player 2

		B	S
Player 1	B	2, 1	0, 0
	S	0, 0	1, 2

- What if the two players make decisions **sequentially** rather than simultaneously?
  - What will they do in equilibrium?
  - How do their payoffs change?
  - Is it better to be the **leader** or the **follower**?

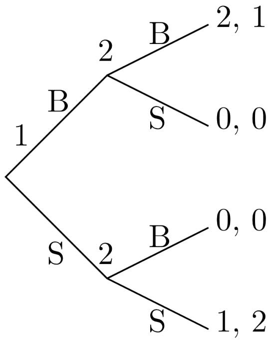
Game tree for dynamic games

- Suppose player 1 **moves** first.
- Instead of a game matrix, the game can now be described by a **game tree**.
  - At each internal node, the label shows who is making a decision.
  - At each link, the label shows an action.
  - At each leaf, the numbers show the payoffs.
- The games is played from the root to leaves.



Optimal strategies

- How should player 1 move?
- She must **predict** how player 2 will response:
  - If B has been chosen, choose B.
  - If S has been chosen, choose S.
- This is player 2's **best response**.
- Player 1 can now make her decision:
  - If I choose B, I will end up with 2.
  - If I choose S, I will end up with 1.
- So player 1 will choose B.
- An **equilibrium outcome** is a "path" goes from the root to a leaf.
  - In equilibrium, they play (B, B).



Sequential moves vs. simultaneous moves

- In the static version, there are two pure-strategy Nash equilibria:
  - (B, B) and (S, S).
- When the game is played dynamically with player 1 moves first, there is only one **equilibrium outcome**:
  - (B, B).
- Their **equilibrium behaviors** change. Is it always the case?
- Being the leader is beneficial. Is it always the case?

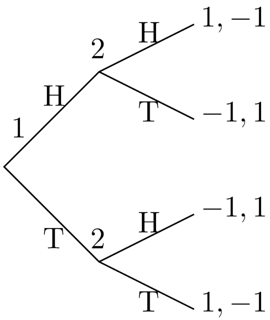
Dynamic matching pennies

- Suppose the game "matching pennies" is played dynamically:

Player 2

		H	T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

- What is the equilibrium outcome?
- There are multiple possible outcomes.
- Being the leader **hurts** player 1.



Backward induction

- In the previous two examples, there are a leader and a follower.
- Before the leader can make her decision, she must anticipate what the follower will do.
- When there are multiple **stages** in a dynamic game, we generally analyze those decision problems **from the last stage**.
  - The second last stage problem can be solved by having the last stage behavior in mind.
  - Then the third last stage, the fourth last stage, ...
- In general, we move **backwards** until the first stage problem is solved.
- This solution concept is called **backward induction**.

Road map

- Prisoners' dilemma.
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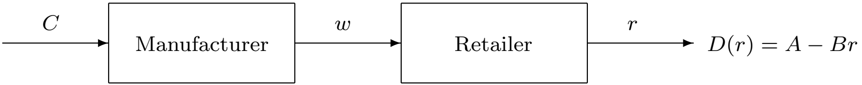
- **Pricing in a supply chain**.

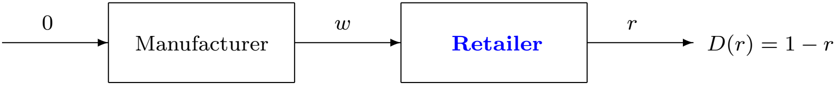
Pricing in a supply chain

- There is a manufacturer and a retailer in a supply chain.



  - The manufacturer supplies to the retailer, who then sells to consumers.
  - The manufacturer sets the **wholesale price** w and then the retailer sets the **retail price** r.
- The demand is $D(r) = A - Br$, where A and B are known constants.
- The unit production cost is C, a known constant.
- Each of them wants to maximize her or his profit.

Pricing in a supply chain (illustrative)

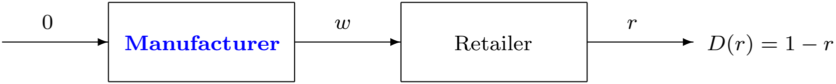


- Let's assume $A = B = 1$ and $C = 0$ for a while.
- Let's apply backward induction to **solve** this game.
- For the retailer, the wholesale price is **given**. He solves

$$\max_{r \geq 0} \ (r - w)(1 - r).$$

- The optimal solution (best response) is $r^*(w) \equiv \frac{w+1}{2}$.

Pricing in a supply chain (illustrative)



- The manufacturer **predicts** the retailer's decision:
  - Given her offer w, the retail price will be $r^*(w) \equiv \frac{w+1}{2}$.
  - More importantly, the **order quantity** (which is the demand) will be

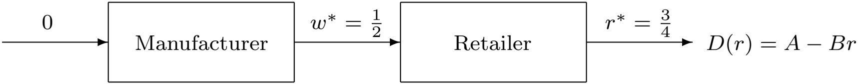
$$1 - r^*(w) = 1 - \frac{w+1}{2} = \frac{1-w}{2}.$$

  - The manufacturer's solves

$$\max_{w \geq 0} \; w\left(\frac{1-w}{2}\right).$$

  - The optimal solution is $w^* = \frac{1}{2}$.

Pricing in a supply chain (illustrative)



- As the manufacturer offers $w^* = \frac{1}{2}$, the resulting retail price is

$$r^* \equiv r^*(w^*) = \frac{w^* + 1}{2} = \frac{3}{4} > \frac{1}{2} = w^*.$$

  - A common practice called **markup**.
- The **sales volume** is $D(r^*) = 1 - r^* = \frac{1}{4}$.
- The retailer earns $(r^* - w^*)D(r^*) = (\frac{1}{4})(\frac{1}{4}) = \frac{1}{16}$.
- The manufacturer earns $w^* D(r^*) = (\frac{1}{2})(\frac{1}{4}) = \frac{1}{8}$.
- In total, they earn $\frac{1}{16} + \frac{1}{8} = \frac{3}{16}$.

Pricing in a supply chain (general)

- For the retailer, the wholesale price is given. He solves

$$\max_{r \geq 0} (r - w)(A - Br)$$

- The optimal solution is $r^*(w) \equiv \frac{Bw+A}{2B}$.
- The manufacturer predicts the retailer's decision:
  - Given her offer w, the retail price will be $r^*(w) \equiv \frac{Bw+A}{2B}$.
  - More importantly, the order quantity (which is the demand) will be $A - Br^*(w) = A - \frac{Bw+A}{2} = \frac{A-Bw}{2}$.
- The manufacturer's problem:

$$\max_{w \geq 0} (w - C)\left(\frac{A - Bw}{2}\right)$$

- The optimal solution is $w^* = \frac{BC+A}{2B}$.

Pricing in a supply chain (general)

- As the manufacturer offers $w^* = \frac{BC+A}{2B}$, the resulting retail price is $r^* \equiv r^*(w^*) = \frac{Bw^*+A}{2B} = \frac{BC+3A}{4B}$.
- The sales volume is $D(r^*) = A - Br^* = \frac{A-BC}{4}$.
- The retailer earns $(r^* - w^*)D(r^*) = (\frac{A-BC}{4B})(\frac{A-BC}{4}) = \frac{(A-BC)^2}{16B}$.
- The manufacturer earns $(w^* - C)D(r^*) = (\frac{A-BC}{2B})(\frac{A-BC}{4}) = \frac{(A-BC)^2}{8B}$.
- In total, they earn $\frac{(A-BC)^2}{16B} + \frac{(A-BC)^2}{8B} = \frac{3(A-BC)^2}{16B}$.

Pricing in a cooperative supply chain

- Suppose the two firms are **cooperative**.
- They decide the wholesale and retail prices together.
- Is there a way to allow both players to be **better off**?
- Consider the following proposal:
  - Let's set $w^{\mathrm{FB}} = C = 0$ and $r^{\mathrm{FB}} = \frac{1}{2}$ (FB: **first best**).
  - The sales volume is
  $$D(r^{\mathrm{FB}}) = 1 - \frac{1}{2} = \frac{1}{2}.$$
  - The total profit is
  $$r^{FB} D(r^{\mathrm{FB}}) = \frac{1}{4}.$$
  - This is **larger** than $\frac{3}{16}$, the total profit generated under decentralization.
- How to split the pie to get a **win-win** situation?