Road map

- Introduction.
- The EOQ model.
- Variants of the EOQ model.
- **The newsvendor model**.

Newsvendor model

- In some situations, people sell **perishable products**.
  - They become valueless after the **selling season** is end.
  - E.g., newspapers become valueless after each day.
  - High-tech goods become valueless once the next generation is offered.
  - Fashion goods become valueless when they become out of fashion.
- In many cases, the seller only have **one chance** for replenishment.
  - E.g., a small newspaper seller can order only once and obtain those newspapers only at the morning of each day.
- Often sellers of perishable products face **uncertain demands**.
- **How many** products one should prepare for the selling season?
  - Not too many and not too few!

Newsvendor model

- Let D be the uncertain demand (so D is a **random variable**).
- Let F and f be the cdf and pdf of D (assuming D is continuous).
  - If D is uniformly distributed between 0 and 100, we have $f(x) = \frac{1}{100}$ and $F(x) = \Pr(D \leq x) = \frac{x}{100}$ for all $x \in [0, 100]$.
  - If D is normally distributed with mean 50 and standard deviation 10:

Overage and underage costs

- Let c_o be the **overage cost** and c_u be the **underage cost**.
  - They are also called overstocking and understocking costs.
  - They are the costs for preparing too many or too few products.
- Components of overage and underage costs may include:
  - Sales revenue r for each unit sold.
  - Purchasing cost c for each unit purchased.
  - Salvage value v for each unit unsold.
  - Disposal fee d for each unit unsold.
  - Shortage cost (loss of goodwill) s for each unit of shortage.
- With these quantities, we have
  - The overage cost $c_o = c + d - v$.
  - The underage cost $c_u = r - c + s$.
- What is an optimal order quantity?
  - As demands are uncertain, we try to minimize the **expected** total overage and underage costs.

Formulation of the newsvendor problem

- Let q be the order quantity (inventory level).
- Let x be the **realization** of demand.
  - D is a random variable and x is a realized value of D.
- Then the realized overage or underage cost is

$$c(q,x) = \begin{cases} c_o(q-x) & \text{if } q \geq x \\ c_u(x-q) & \text{if } q < x \end{cases}$$

  or simply $c(q,x) = c_o(q-x)^+ + c_u(x-q)^+$, where $y^+ = \max(y,0)$.
- Therefore, the **expected total cost** is

$$c(q,D) = \mathbb{E}\Big[c_o(q-D)^+ + c_u(D-q)^+\Big].$$

- We want to find a quantity q that solves the NLP

$$\min_{q \geq 0} \ \mathbb{E}\Big[c_o(q-d)^+ + c_u(d-q)^+\Big].$$

Convexity of the cost function

- The cost function $c(q, D) = \mathbb{E}\Big[c_o(q - D)^+ + c_u(D - q)^+\Big]$.
- By assuming that D is continuous, the cost function $c(q, D)$ is

$$
\begin{aligned}
& \int_0^\infty \Big[c_o(q-x)^+ + c_u(x-q)^+\Big] f(x)dx \\
= & \int_0^q \Big[c_o(q-x) + c_u \cdot 0\Big] f(x)dx + \int_q^\infty \Big[c_o \cdot 0 + c_u(x-q)\Big] f(x)dx \\
= & \; c_o \int_0^q (q-x)f(x)dx + c_u \int_q^\infty (x-q)f(x)dx \\
= & \; c_o\left[q\int_0^q f(x)dx - \int_0^q xf(x)dx\right] + c_u\left[\int_q^\infty xf(x)dx - q\int_q^\infty f(x)dx\right] \\
= & \; c_o\left[qF(q) - \int_0^q xf(x)dx\right] + c_u\left[\int_q^\infty xf(x)dx - q\big(1 - F(q)\big)\right].
\end{aligned}
$$

Convexity of the cost function

- We have

$$c(q, D) = c_o\left[qF(q) - \int_0^q xf(x)dx\right] + c_u\left[\int_q^\infty xf(x)dx - q\big(1 - F(q)\big)\right].$$

- The first-order derivative of $c(q, D)$ is

$$\begin{aligned} & c'(q, D) \\ = & c_o\Big[F(q) + qf(q) - qf(q)\Big] + c_u\Big[-qf(q) - (1 - F(q)) + qf(q)\Big] \\ = & c_o\big[F(q)\big] - c_u\big[1 - F(q)\big]. \end{aligned}$$

- The second-order derivative of $c(q, D)$ is

$$c''(q, D) = c_o f(q) + c_u f(q) = f(q)(c_u + c_o) > 0.$$

- So $c(q, D)$ is convex in q.

Optimizing the order quantity

- Let q^* be the order quantity that satisfies the FOC, we have

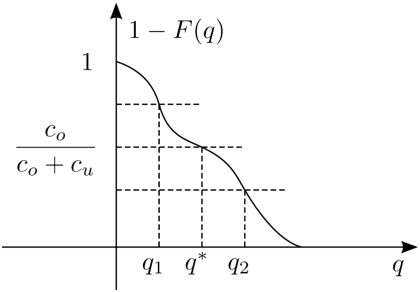
$$\begin{aligned} c_o F(q^*) - c_u(1 - F(q^*)) = 0 \\ \Rightarrow F(q^*) = \frac{c_u}{c_o + c_u} \quad \text{or} \quad 1 - F(q^*) = \frac{c_o}{c_o + c_u}. \end{aligned}$$

- Such q^* must be positive (for regular demand distributions).
  - So q^* is optimal.
  - The quantity q^* is called the **newsvendor** quantity.
  - Note that the only assumption we made is that D is continuous!
- Note that to minimize the expected total cost, the seller should **intentionally** create some shortage!
  - The optimal probability of having a shortage is $1 - F(q^*) = \frac{c_o}{c_o + c_u}$.

Interpretations of the newsvendor quantity

- The probability of having a shortage, $1 - F(q)$, is decreasing in q.
- The newsvendor quantity q^* satisfies $1 - F(q^*) = \frac{c_o}{c_o + c_u}$.
- The optimal quantity q^* is:
  - Decreasing in c_o.
  - Increasing in c_u.

  Why?



Example 1

- Suppose for a newspaper:
  - The unit purchasing cost is \$5.
  - The unit retail price is \$15.
  - The demand is uniformly distributed between 20 to 50.
- Overage cost $c_o = 5$ and underage cost $c_u = 15 - 5 = 10$.
- The optimal order quantity q^* satisfies

$$1 - F(q^*) = \left(1 - \frac{q^* - 20}{50 - 20}\right) = \frac{5}{5 + 10} \quad \Rightarrow \quad \frac{50 - q^*}{30} = \frac{1}{3},$$

  which implies $q^* = 40$.
- If the unit purchasing cost decreases to \$4, we need $\frac{50-q^{**}}{30} = \frac{4}{15}$ and thus $q^{**} = 42$.
  - As the purchasing cost decreases, we **prefer overstocking** more. Therefore, we stock more.

Example 2

- Suppose for one kind of apple:
  - The unit purchasing cost is \$15, the unit retail price is \$21, and the unit salvage value is \$1.
  - The demand is normally distributed with mean 90 and standard deviation 20.
  - Overage cost $c_o = 15 - 1 = 14$ and underage cost $c_u = 21 - 15 = 6$.
- The optimal order quantity q^* satisfies

$$\Pr(D < q^*) = \frac{6}{14+6} \quad \Rightarrow \quad \Pr\left(Z < \frac{q^* - 90}{20}\right) = 0.3,$$

  where $Z \sim \text{ND}(0,1)$.
- By looking at a probability table or using a software, we find $\Pr(Z < -0.5244) = 0.3$. Therefore, $\frac{q^*-90}{20} = -0.5244$ and $q^* = 79.512$.
  - As the purchasing cost is so high, we want to **reject more than half** of the consumers!