Linear Algebra and its Applications, Spring 2013
Midterm Exam

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Note 1. You DO NOT need to return the problem sheet.

Note 2. In total there are 110 points but you may get at most 100.

1. (20 points; 5 points each) Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 5 & 1 & 0 & 4 \\ 1 & 2 & 4 & 0 & -3 & 4 \\ 3 & 6 & 14 & 4 & -3 & 13 \\ 2 & 4 & 10 & 0 & 0 & 11 \end{bmatrix}.$$

   (a) Do the LU decomposition for A. Perform row exchanges and write down the P matrix when necessary. Do the LU decomposition, not the LDU decomposition.
   (b) Solve $Ax = b$, where $b = (1, 0, 0, 0)$. Find all the solutions if there are multiple. If there is no solution, show it.
   (c) Find the four fundamental subspaces of A.
   (d) Project $d = (3, 2, 4, 5)$ onto the column space of A.

2. (10 points; 5 points each) Consider

$$A_1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & 5 \\ 0 & 2 & 3 & 0 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & x & 5 \\ 0 & 2 & 3 & 0 \end{bmatrix}.$$

   (a) Find the determinant of A_1.
   (b) Find the value(s) of x such that the dimension of the left nullspace of A_2 is greater than 0.

3. (10 points; 5 points each) Consider the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 \end{bmatrix}.$$

   (a) Find a right inverse of A if there is one or show that there is none.
   (b) Find a left inverse of A if there is one or show that there is none.

4. (10 points) A set S is a convex set if for any x_1, $x_2 \in S$ and any $\lambda \in [0, 1]$, $\lambda x_1 + (1 - \lambda)x_2 \in S$. According to this definition, to prove or disprove the following statements.

   (a) (3 points) The column space of any matrix A is a convex set.
   (b) (3 points) The set of 3×3 symmetric matrices is a convex set.
   (c) (4 points) The set of 3×3 singular matrices is a convex set.

5. (15 points; 5 points each) Suppose matrix A's columns are a_1, ..., a_n and matrix C's columns are a_1, ..., a_n, and c, i.e.,

$$A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} | & & | & | \\ a_1 & \cdots & a_n & c \\ | & & | & | \end{bmatrix}.$$

   For each subproblem below, two subspaces are given. Find out which one is contained in the other one or show that no one contains the other one.

(a) The column space of A and the column space of C.
(b) The nullspace of A and the nullspace of C.
(c) The left nullspace of A and the left nullspace of C.

6. (10 points) Consider a vector $a \in \mathbb{R}^m$ and another vector $b \in \mathbb{R}^n$.

   (a) (2 points) Find the dimension of the matrix ab^T.
   (b) (2 points) Find the rank of the matrix ab^T.
   (c) (2 points) Find the set of vectors c such that $ab^T x = c$ is solvable.
   (d) (4 points) Let subspace $V = \{x \in \mathbb{R}^m | x = ka, k \in \mathbb{R}\}$ and subspace $W = \{y \in \mathbb{R}^n | y = kb, k \in \mathbb{R}\}$. What is the dimension of the matrix A that performs a linear transformation from V to W? Prove or disprove that this linear transformation invertible. If it is, prove or disprove that A^{-1} transforms vectors from W back to V.

7. (10 points; 5 points each) Consider two vectors $a = (3, 1, 2)$ and $b = (-1, 1, 1)$.

   (a) Apply the Gram-Schumidt process on $c = (1, 1, 0)$ to find a unit vector $q = (q_1, q_2, q_3)$ that is orthogonal to both a and b. Please note that your answer MUST come from c. You CANNOT find an arbitrary vector, even if it is orthogonal to a and b!
   (b) Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \\ q_1 & q_2 & q_3 \end{bmatrix}.$$

   Find the determinant of AA^T.

8. (5 points) Given any matrix A, prove or disprove that the nullspace of A is the same as the nullspace of $A^T A$.

9. (10 points) Consider the following transformations. For each of them, prove or disprove that it is a linear transformation. If it is, find a matrix A that performs the linear transformation.

   (a) (3 points) The transformation on $\mathbb{R}^2$ that maps a vector (x, y) to $(x + y, 0)$.
   (b) (3 points) The transformation that maps a square matrix to its transpose.
   (c) (4 points) The transformation that maps a matrix $A \in \mathbb{R}^{2\times 2}$ to $(d_1, d_2) \in \mathbb{R}^2$, where d_1 and d_2 are the pivots of A. As an example, if we call the transformation T, then

$$T\left(\begin{bmatrix} 3 & 1 \\ 6 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

10. (10 points; 5 points each) Solve the following two problems:

   (a) Find the inverse of

$$\begin{bmatrix} 1 & 2 & 2 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   or show that the inverse does not exist.
   (b) Find the least square solution for $Ax = b$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.$$