# Linear Algebra and its Applications, Spring 2013
# Homework 14

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**Note 1.** This homework is due ***8:30 am, December 24, 2013***. Please submit a hard copy into the homework box outside the TAs' lab.

**Note 2.** "Problem sets" should be found in the textbook (the fourth edition).

1. (20 points) In this problem, we derive a necessary condition for local minima of constrained nonlinear programs. We consider a problem of minimizing $f(x)$ over $x \in F \subseteq \mathbb{R}^n$.

   (a) For a set $F \subseteq \mathbb{R}^n$, an interior point $x \in F$ is a point that does not lie on the boundary. More precisely, if there exists a small enough $\epsilon > 0$ such that $x + \lambda d \in F$ for all unit-length $d \in \mathbb{R}^n$ and $\lambda \in [0, \epsilon)$, then $x$ is an interior point. Such a unit-length direction $d$ is called a feasible direction at $x$. According to this definition, list three interior points of $S = \{x \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_3^2 \leq 1\}$.

   (b) For a set $F \subseteq \mathbb{R}^n$, a point $x \in F$ is a boundary point if it is not an interior point. For the $S$ defined in Part (a), what are the boundary points?

   **Note.** An extreme point is a boundary point but the converse is not true.

   (c) Show that if an interior point $\bar{x} \in F$ is a local minimum of $f(\cdot)$, we must have $\nabla f(\bar{x}) = 0$.

   (d) Show that if a boundary point $\bar{x} \in F$ is a local minimum of $f(\cdot)$, we must have $\nabla f(\bar{x})^T d \geq 0$ for all feasible direction $d$ at $\bar{x}$.

   **Note.** Intuitively, along any feasible direction, the function must "go up".

   **Note.** The general necessary condition is: If a point $\bar{x} \in F$ is a local minimum of $f(\cdot)$, we must have $\nabla f(\bar{x})^T d \geq 0$ for all feasible direction $d$ at $\bar{x}$. Why we do not need to assume that $\bar{x}$ is a boundary point?

2. (20 points) Consider the nonlinear program

$$\begin{aligned} \min \quad & -(p-c)(a-bp) \\ \text{s.t.} \quad & p \geq 0 \end{aligned}$$

   in which a seller chooses a price $p \in \mathbb{R}$ to maximize its profit. $a > 0$, $b > 0$, and $c \geq 0$ are parameters.

   (a) (5 points) Show that the objective function is convex over $\mathbb{R}$. Is it strictly convex?

   (b) (10 points) Given that the feasible region is convex, this is a convex program. Use this fact to solve the program by finding a global minimum $p^*$ as a function of $a$, $b$, and $c$.

   **Note.** Do you need to take boundary points into consideration?

   (c) (5 points) Let $a = b = 1$ and $c = 0$, depict the objective function.

3. (20 points) Consider the nonlinear program

$$\begin{aligned} \min \quad & -(a-2d)^2 d \\ \text{s.t.} \quad & 0 \leq d \leq \frac{a}{2} \end{aligned}$$

   in which one looks for a length $x \in \mathbb{R}$ to maximize the volume of the box made by folding a paper. $a > 0$ is a parameter.

   (a) (5 points) Show that the objective function is not convex over the feasible region.

   **Note.** It is NOT enough to show that the objective function is not convex over $\mathbb{R}$!

(b) (10 points) Find all the stationary points of the objective function. Are they all feasible? For each of them, determine whether it is a local minimum, local maximum, or a saddle point. Then determine the global minimum $d^*$.

**Note.** Do you need to take boundary points into consideration?

(c) (5 points) Let $a = 6$, depict the objective function.

4. (20 points) Consider the nonlinear program

$$\begin{aligned} \min \quad & \|x - x^0\| \\ \text{s.t.} \quad & \|x - x^i\| \le d \quad \forall i = 1, ..., m. \end{aligned} \tag{1}$$

in which one chooses a location $x = (x_1, x_2) \in \mathbb{R}^2$ to locate an airport to minimize the distance to the capital at $x_0$ while ensuring the distance between the airport and each city is within $d$. $x^i = (x_1^i, x_2^i) \in \mathbb{R}^2$ for $i = 0, ..., m$ and $d > 0$ are given parameters.

(a) (5 points) Show that the program is equivalent to

$$\begin{aligned} \min \quad & \|x - x^0\|^2 \\ \text{s.t.} \quad & \|x - x^i\|^2 \le d^2 \quad \forall i = 1, ..., m. \end{aligned} \tag{2}$$

(b) (5 points) Show that the objective function of the program in (2) is convex over $\mathbb{R}^2$.

(c) (5 points) Show that the feasible region of the program in (2) is convex.

**Note.** Now we know that the program in (2) is a convex program. Though we are unable to analytically solve the program as we did for Problems 2 and 3, a greedy search algorithm can numerically solve it.

(d) (5 points) Suppose the feasible region is nonempty. Will an optimal solution be a boundary point of the feasible region?

**Note.** It can be shown that the program in (1) is not a convex program. In general, a nonconvex program can be equivalent to a convex program and solving the latter is much easier.

5. (20 points) Solve the following problems:

(a) (5 points) Show that $\{(x_1, x_2) \in (0, \infty) \times \mathbb{R} | x_2 \le \ln x_1\}$ is convex.

(b) (5 points) Find a set $F \subseteq \mathbb{R}^3$ over which $f_1(x_1, x_2, x_3) = x_1^3 + 2x_1x_3 + x_2x_3 - x_3^2$ is convex.

(c) (10 points) For $f_2(x_1, x_2, x_3) = x_1^3 - x_1x_2 + 2x_1x_3 + x_2x_3 - x_3^2$, find all the stationary points. For each of them, determine whether it is a local minimum.