Operations Research, Spring 2013
Homework 01

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. (Modified from Problem 2.3.6; 5 points) Consider a linear system

$$\begin{array}{ccccccc} & & 2x_2 & + & 2x_3 & = & 4 \\ x_1 & + & 2x_2 & + & x_3 & = & 4 \\ & & x_2 & - & x_3 & = & 0 \end{array}.$$

   Use Gauss-Jordan elimination to determine whether the system has a unique solution, infinitely many solutions, or no solution. In the first two cases, write down the solution(s) explicitly.

2. (Modified from Problem 2.5.2; 5 points) Use Gauss-Jordan elimination to find the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}.$$

3. (10 points; 2 points each) For each of the following sets, determine whether it is a convex set. You need to at least intuitively explain why.

   (a) $S_1 = \{x \in \mathbb{R} | x^2 \geq 4\}$.[1]

   (b) $S_2 = \{(x, y) \in \mathbb{R}^2 | x + 2y \leq 6, 2x + y \leq 6\}$.

   (c) $S_3 = \{(x, y) \in \mathbb{R}^2 | y \geq \ln x, x > 0\}$.

   (d) $S_4 = \{(x, y) \in \mathbb{R}^2 | \frac{x^2}{9} + \frac{y^2}{4} \leq 1, x + y \geq 1, x \geq 0, y \geq 0\}$.

   (e) $S_5 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1 + 2x_2 + 3x_3 \geq 6, x_1 \leq 6, x_2 \leq 3, x_3 \leq 2\}$.

4. (10 points; 2 points each) For each of the following functions, determine whether it is a convex function over the given domain. You need to at least intuitively explain why.

   (a) $f_1(x) = x^2 \ \forall x \in \mathbb{R}$.

   (b) $f_2(x) = x^3 \ \forall x \in \mathbb{R}$.

   (c) $f_3(x) = \begin{cases} \frac{1}{2}x & \text{if } x < 4 \\ 2x & \text{if } x \geq 4 \end{cases} \quad \forall x \in [0, \infty)$.

   (d) $f_4(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases} \quad \forall x \in \mathbb{R}$.

   (e) $f_5(x, y) = 2x + y \ \forall (x, y) \in [0, \infty) \times [0, \infty)$.[2]

5. (10 points) Consider the following linear program

$$\begin{array}{rcccccccc} \max & x_1 & + & 2x_2 & - & 3x_3 & & \\ \text{s.t.} & 6x_1 & + & 8x_2 & - & 4x_3 & \leq & 10 \\ & 5x_1 & - & 3x_2 & - & 2x_3 & \geq & 7 \\ & x_1 & + & & + & 2x_3 & = & 4 \\ & x_1 & & & & & \geq & 0. \end{array}$$

   (a) (6 points) Express the linear program into the matrix representation min cx s.t. $Ax \leq b$, where c is a row vector, b and x are two column vectors, and A is a matrix.

[1] This means the set of all real number x that satisfy $x^2 \geq 4$.
[2] This means the domain is the first quadrant.

(b) (2 points) Is $(x_1, x_2, x_3) = (2, 0, 1)$ feasible to the linear program? Why? What objective value does this solution correspond to?

(c) (2 points) Is $(x_1, x_2, x_3) = (1, 1, 2)$ feasible to the linear program? Why? What objective value does this solution correspond to?

6. (Modified from Problem 3.1.1; 10 points) Farmer Jones must determine how many acres of corn and wheat to plant this year. An acre of wheat yields 25 bushels of wheat and requires 10 hours of labor per week. An acre of corn yields 10 bushels of corn and requires 4 hours of labor per week. All wheat can be sold at $4 per bushel, and all corn can be sold at $3 per bushel. Seven acres of land and 40 hours per week of labor are available. Government regulations require that at least 30 bushels of corn be produced during the current year. Let x_1 be the number of acres of corn planted, and x_2 be number of acres of wheat planted. Using these decision variables, formulate an LP whose solution will tell Farmer Jones how to maximize the total revenue from wheat and corn.

7. (Modified from Problem 3.4.2; 10 points) U.S. Labs manufactures mechanical heart valves from the heart valves of pigs. Different heart operations require valves of different sizes. U.S. Labs purchases pig valves from three different suppliers. The cost and size mix of the valves purchased from each supplier are given in the table below. Each month, U.S. Labs places one order with each supplier. At least 500 large, 300 medium, and 300 small valves must be purchased each month. Because of limited availability of pig valves, at most 700 valves per month can be purchased from each supplier. Formulate an LP that can be used to minimize the cost of acquiring the needed valves.

Supplier	Cost per Valve ($)	Large (%)	Medium (%)	Small (%)
1	5	40	40	20
2	4	30	35	35
3	3	20	20	60

8. (Modified from Problem 3.5.2; 10 points) During each 4-hour period, the Smalltown police force requires the following number of on-duty police officers: 12 midnight to 4 A.M. – 8; 4 to 8 A.M. – 7; 8 A.M. to 12 noon – 6; 12 noon to 4 P.M. – 6; 4 to 8 P.M. – 5; 8 P.M. to 12 midnight – 4. Each police officer works two consecutive 4-hour shifts. Formulate an LP that can be used to minimize the number of police officers needed to meet Smalltown's daily requirements.

9. (Modified from Problem 3.8.7; 10 points) Eli Daisy uses chemicals 1 and 2 to produce two drugs. Drug 1 must be at least 70% chemical 1, and drug 2 must be at least 60% chemical 2. Up to 40 oz of drug 1 can be sold at $6 per oz; up to 30 oz of drug 2 can be sold at $5 per oz. Up to 45 oz of chemical 1 can be purchased at $6 per oz, and up to 40 oz of chemical 2 can be purchased at $4 per oz. Formulate an LP that can be used to maximize Daisy's profits.

10. (Modified from Problem 3.10.1; 10 points) A customer requires during the next four months, respectively, 50, 65, 100, and 70 units of a commodity (no backlogging is allowed). Production costs are $5, $8, $4, and $7 per unit during these months. The storage cost from one month to the next is $2 per unit (assessed on ending inventory). It is estimated that each unit on hand at the end of month 4 could be sold for $6. Formulate an LP that will minimize the net cost incurred in meeting the demands of the next four months.

11. (Modified from Problem 3.10.3; 10 points) James Beerd bakes cheesecakes and Black Forest cakes. During any month, he can bake at most 65 cakes. The costs per cake and the demands for cakes, which must be met on time, are listed below. It costs 50 cents to hold a cheesecake, and 40 cents to hold a Black Forest cake, in inventory for a month. Formulate an LP that can minimize the total cost of meeting the next three months' demands.

Item	Month 1 Demand	Month 1 Cost/Cake ($)	Month 2 Demand	Month 2 Cost/Cake ($)	Month 3 Demand	Month 3 Cost/Cake ($)
Cheesecake	40	3.00	30	3.40	20	3.80
Black Forest	20	2.50	30	2.80	10	3.40