Operations Research, Spring 2013
Homework 04

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1. (Modified from Problem 4.5.3; 30 points; 5 points each) Consider the following linear program

$$\begin{aligned} z^* = \max \quad & 2x_1 - x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + x_2 + x_3 \leq 60 \\ & x_1 - x_2 + 2x_3 \leq 10 \\ & x_1 + x_2 - x_3 \leq 20 \\ & x_i \geq 0 \quad \forall\, i = 1, ..., 3. \end{aligned}$$

   (a) Find the standard form.

   (b) Given a basis $B = \{x_4, x_5, x_6\}$ and a set of nonbasic variables $N = \{x_1, x_2, x_3\}$, find $A_B^{-1}A_N$, $A_B^{-1}b$, $c_B A_B^{-1}b$, and $c_B A_B^{-1}A_N - c_N$.

   (c) Prepare the initial tableau for this program. In the tableau, where may you find the four quantities calculated in Part (b)?

   (d) Given another basis $B = \{x_4, x_1, x_6\}$ and another set of nonbasic variables $N = \{x_5, x_2, x_3\}$, find $A_B^{-1}A_N$, $A_B^{-1}b$, $c_B A_B^{-1}b$, and $c_B A_B^{-1}A_N - c_N$.
   Hint. Do not forget that the orders of variables in B and N matters!

   (e) Starting from the initial tableau you get in Part (c), do one iteration by entering x_1. In your second tableau, where may you find the four quantities calculated in Part (d)?

   (f) Starting from the second tableau you get in Part (e), complete the simplex method and find the optimal solution to the original program $x^* = (x_1^*, x_2^*, x_3^*)$ and the corresponding objective value z^*.

2. (Modified from Problem 4.6.1; 20 points; 10 points each) Consider the following linear program

$$\begin{aligned} z^* = \min \quad & 4x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \leq 8 \\ & -x_2 \leq 5 \\ & x_1 + x_2 \leq 4 \\ & x_1 \geq 0, x_2 \leq 0. \end{aligned}$$

   (a) Use the simplex method to solve this program. Find an optimal solution $x^* = (x_1^*, x_2^*)$ and the corresponding objective value z^*.

   (b) For each basis/tableau you encounter in Part (a), write down the associated basic feasible solution. For these basic feasible solutions, graphically indicate where the corresponding solutions to the original program are. Then illustrate the route you go through when solving this program.
   Note. Though not required, you may certainly want to apply the graphical approach to verify your solution.

3. (Modified Problem 4.5.2; 20 points; 10 points each) Consider the linear program we discussed in class

$$\begin{aligned} z^* = \min \quad & -2x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_i \geq 0 \quad \forall\, i = 1, 2. \end{aligned}$$

The optimal solution we found in class was $(x_1^*, x_2^*) = (\frac{10}{3}, \frac{4}{3})$. Note that in the first iteration, we entered x_1.

(a) Use the simplex method to solve this program. However, enter x_2 in the first iteration.

(b) Graphically depict the route you go through when solving this program in Part (a).

Note. This program provides one example that if a linear program has a unique optimal solution, different routes (due to entering different variables) all lead to the same unique optimal solution. This can be proved to be true in general. Nevertheless, the selection rule does affect something, especially when the linear program is degenerate. We will try to discuss this issue in class sooner or later.

4. (Modified from Problem 3.Review.47; 15 points) The Gotham City Police Department employs 30 police officers. Each officer works 5 days per week. The crime rate fluctuates with the day of the week, so the number of police officers required each day depends on which day of the week it is: Saturday, 28; Sunday, 18; Monday, 18; Tuesday, 24; Wednesday, 25; Thursday, 16; Friday, 21. The police department wants to schedule police officers to minimize the number whose days off are NOT consecutive. Formulate an LP that will accomplish this goal.

   Hint. Seven decision variable are not enough because there are more than seven ways to assign an officer to five unnecessarily consecutive days. How many possible ways do you have?

5. (15 points) Consider a linear program

$$
\begin{aligned}
\min \quad & cx \\
\text{s.t.} \quad & Ax \leq b \\
& x \geq 0.
\end{aligned}
$$

   Let the number of constraints be m and the number of variables be n.

   (a) (5 points) To get the standard form, how many slack variables do you need?

   (b) (10 points) Suppose $b \geq 0$. If you select all the slack variables to be the basis, the resulting basic solution is always feasible. Why?

   Note. For a linear program with nonnegative RHS, nonnegative variables, and only no-greater-than functional constraints, choosing the slack variables to form an initial basis for the simplex method always works. Beside this way, is there any easy way of forming an initial basis? If we want to solve a linear program not in this format, how to find an initial basis? This issue will be discussed sooner or later.