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Suggested Solution for Homework 06

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1. The branch-and-bound tree for solving this problem is depicted in Figure 1. The optimal solution, as we expect, is still $(5, 0)$. The optimal objective value is 40.

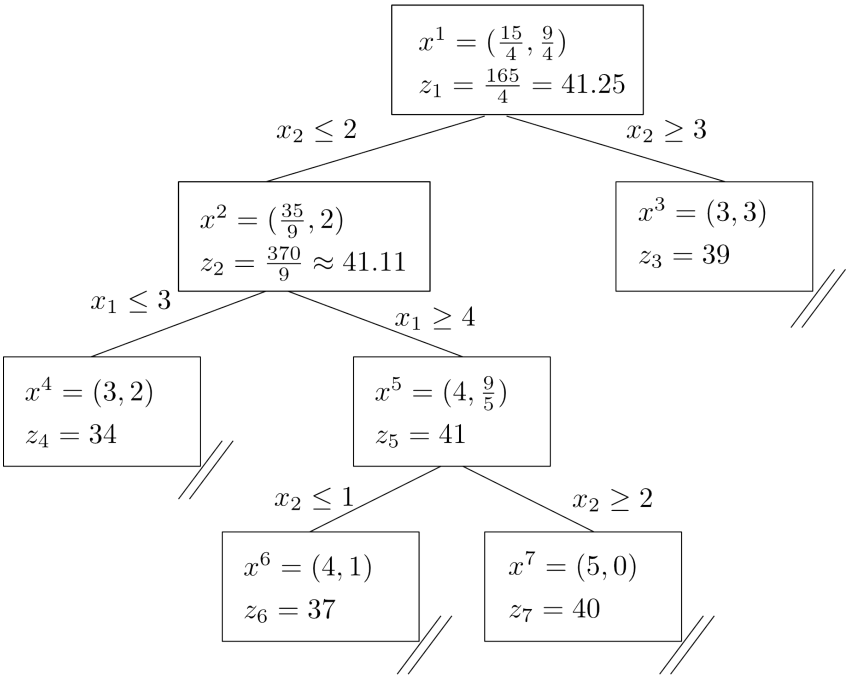
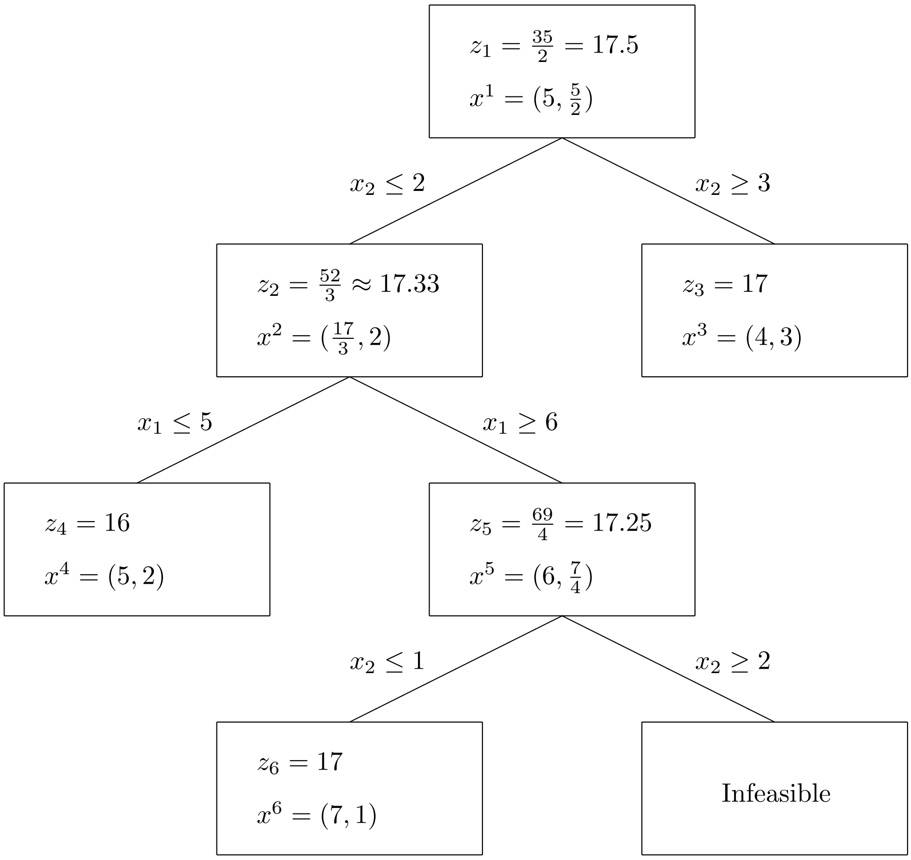


Figure 1: Branch-and-bound tree for Problem 1

2. A branch-and-bound tree is depicted below. For each node, we need to solve a linear program (probably with some additional constraints). For these two-dimensional problems, it is suggested to use the graphical approach to solve them. The complete process is summarized below:

   - In node 1, we solve the LP relaxation of the original IP. Because x_2 is fractional, we branch on x_2.
   - In node 2, the optimal solution has x_1 fractional. We keep this node on hold for a moment.
   - In node 3, the optimal solution is an integer solution. Because $z_2 > z_3$, it is possible to find an integer solution better than x^3 by going further from node 2. Therefore, we branch on x_1 from node 2.
   - In node 4, the optimal solution is an integer solution. Because it is not better x^3, it can be ignored.
   - In node 5, the optimal solution has x_2 fractional. Because $z_5 > z_3$, it is possible to find an integer solution better than x^3 by going further from node 5. Therefore, we branch x_2 from node 5.
   - In node 6, the optimal solution is an integer solution. Because it is not better x^3, it can be ignored.
   - In node 7, the problem is infeasible.

We thus conclude that an optimal solution is $x^3 = (4, 3)$ with the objective value $z_3 = 17$. Note that all we want is "one" optimal solution.



Remark 1. If you use some other branching rule, it is possible to get $x^6 = (7, 1)$ as your optimal solution. As long as your solution process is reasonable, it is fine.

Remark 2. If you observe that all the objective coefficients are integers and all variables must be integers, you may apply the idea that "the best integer successor's objective value is at most the floor of that of its ancestor". With this in mind, you may stop after examining node 3, because $z_2 < 18$ implies that branching from node 2 will not result in an integer solution better than x^3.

3. Let item 1, 2, ..., and 5 be the bedroom set, the dining room set, the stereo, the sofa, and the TV set. Then we define the decision variable

$$x_i = \begin{cases} 1 & \text{if we choose item } i \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } i = 1, 2, ..., 5$$

and formulate the problem as

$$\begin{array}{rl} \max & 60x_1 + 48x_2 + 14x_3 + 31x_4 + 10x_5 \\ \text{s.t.} & 8x_1 + 6x_2 + 3x_3 + 4x_4 + 2x_5 \leq 11 \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 5. \end{array}$$

We then relax the binary constraint of this problem and get the relaxed problem (P^1)

$$\begin{array}{rl} \max & 60x_1 + 48x_2 + 14x_3 + 31x_4 + 10x_5 \\ \text{s.t.} & 8x_1 + 6x_2 + 3x_3 + 4x_4 + 2x_5 \leq 11 \\ & 0 \leq x_i \leq 1 \quad \forall i = 1, ..., 5. \end{array}$$

Now we are ready to use branch-and-bound to solve it. The complete solution process is listed below.[1]

- (P^1): $(\frac{1}{8}, 1, 0, 1, 0)$; 86.5; continue.
  - $x_1 \leq 0$: (P^2): $(0, 1, 0, 1, \frac{1}{2})$; 84; continue.
    - $x_5 \leq 0$: (P^4): $(0, 1, \frac{1}{3}, 1, 0)$; 83.67; continue.
      - $x_3 \leq 0$: (P^6): $(0, 1, 0, 1, 0)$; 79; current candidate, so stop.
      - $x_3 \geq 1$: (P^7): $(0, 1, 1, \frac{1}{2}, 0)$; 77.5; not good enough, so stop.
    - $x_5 \geq 1$: (P^5): $(0, 1, 0, \frac{3}{4}, 1)$; 81.25, continue.
      - $x_4 \leq 0$: (P^{12}): $(0, 1, 1, 0, 1)$; 72; not good enough, so stop.
      - $x_4 \geq 1$: (P^{13}): $(0, \frac{5}{6}, 0, 1, 1)$; 81; continue.
        - $x_2 \leq 0$: (P^{12}): $(0, 0, 1, 1, 1)$; 55; not good enough, so stop.
        - $x_2 \geq 1$: (P^{13}): infeasible, so stop.
  - $x_1 \geq 1$: (P^3): $(1, \frac{1}{2}, 0, 0, 0)$; 84; continue.
    - $x_2 \leq 0$: (P^8): $(1, 0, 0, \frac{3}{4}, 0)$; 83.25; continue.
      - $x_4 \leq 0$: (P^{10}): $(1, 0, \frac{1}{3}, 0, 1)$; 74.76; not good enough, so stop.
      - $x_4 \geq 1$: (P^{11}): infeasible, so stop.
    - $x_2 \geq 1$: (P^9): infeasible, so stop.

So the optimal solution is found in solving subproblem (P^6). It is to bring the dining room set and the sofa (item 2 and 4). The total values we get is 79. An unrealistic assumption we made is that these items can be transformed into any shape we like.

4. Let New York, Los Angeles, Chicago, and Atlanta be city 1, 2, 3, and 4. We then define

$$x_{ij} = \text{units shipped from city } i \text{ to region } j,\ i = 1, ..., 4,\ j = 1, ..., 3 \text{ and}$$
$$y_i = \begin{cases} 1 & \text{if city } i \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}, \ i = 1, ..., 4$$

as our decision variables. We also denote F_i as the fixed cost for city i, D_j as the demand vector for region j, and C_{ij} as the variable shipping cost between city i and region j. The capacity of each city is K. Let M_is be some very large numbers for a while, the complete formulation is

$$\begin{array}{lll} \min & \sum_{i=1}^{4} \left(F_i y_i + \sum_{j=1}^{3} C_{ij} x_{ij} \right) & \\ \text{s.t.} & \sum_{i=1}^{4} x_{ij} \geq D_j & \forall\ j = 1, ..., 3 \\ & \sum_{j=1}^{3} x_{ij} \leq K & \forall\ i = 1, ..., 4 \\ & \sum_{j=1}^{3} x_{ij} \leq M_i y_i & \forall\ i = 1, ..., 4 \\ & y_1 \leq y_2, \quad \sum_{i=1}^{4} y_i \leq 2, \quad y_2 + y_4 \geq 1 & \\ & x_{ij} \geq 0 & \forall\ i = 1, ..., 4, j = 1, ..., 3 \\ & y_i \in \{0, 1\} & \forall\ i = 1, ..., 4. \end{array}$$

The objective is to minimize the total fixed and variable costs. The first constraint is for us to satisfy all the demands. The second constraint is for capacity. The third one sets the value for y_i's according to the values of x_{ij}'s: y_i should be 1 if $\sum_{j=1}^{3} x_{ij} > 0$. In other words, we need to choose city i as long as we ship anything from it, regardless of the destination. The fourth one makes sure that if we choose New York (city 1) then we choose Los Angeles (city 2). The fifth one allows us to choose at most 2 cities. The sixth one is saying that we must choose either Atlanta or Los Angeles. Nonnegativity and binary constraints follow. If we want to replace M_i with tighter upper bounds, we need to find the upper bounds of $\sum_{j=1}^{3} x_{ij}$ so that we can correctly determine the value of y_i

[1]For each subproblem, we summarize its optimal solution, the corresponding objective value, and whether to branch it. Note that the label of subproblems indicates the sequence of solving them. That is, we solve (P^i) before (P^j) if $i < j$. In general, we adopt the breadth-first-search rather than the depth-first-search.

according to $\sum_{j=1}^{3} x_{ij}$. First, $K = 100$ is an upper bound of $\sum_{j=1}^{3} x_{ij}$ since a city can ship out no more than 100 units. Moreover, we may find another upper bound of $\sum_{j=1}^{3} x_{ij}$ by assuming that only city i is chosen. Since we will ship exactly the total demands $\sum_{j=1}^{3} D_j$ from city i in this case, we know $\sum_{j=1}^{3} D_j$ is an upper bound of $\sum_{j=1}^{3} x_{ij}$. Therefore, $\min\left\{K, \sum_{j=1}^{3} D_j\right\}$ is suggested to replace M_i.[2]

5. (a) Let

$$x_i = \text{production quantity in plant } i,\ i = 1, ..., 3, \text{ and}$$

$$z_i = \begin{cases} 1 & \text{if plant } i \text{ is open} \\ 0 & \text{otherwise} \end{cases},\ i = 1, ..., 3$$

be the decision variables. Also let $F = (80000, 40000, 30000)$ be the vector of fixed costs, $C = (20, 25, 30)$ be the vector of variable costs, $K = (6000, 7000, 6000)$ be the vector of capacity levels. The complete formulation is

$$\begin{array}{lll} \min & \sum_{i=1}^{3} \left(F_i z_i + C_i x_i\right) & \\ \text{s.t.} & \sum_{i=1}^{3} x_i = 12000 & \\ & x_i \leq K_i z_i & \forall\ i = 1, ..., 3 \\ & x_i \geq 0 & \forall\ i = 1, ..., 3 \\ & z_i \in \{0, 1\} & \forall\ i = 1, ..., 3. \end{array}$$

The objective function minimizes the total cost. The first constraint ensures demand fulfillment. The second constraint relates x_i and z_i ($z_i = 1$ if $x_i > 0$) and ensures the production quantity cannot exceed the capacity. The third constraint is the nonnegativity constraint. The last constraint is the binary constraint.

(b) The MS Excel Solver program is contained in the MS Excel file "ORSp13_hw06_sol.xlsx" under the sheet "Problem 5". The optimal solution is to build the first two plants, produce 6000 units in plant 1, and produce 6000 units in plant 2. The associated optimal solution is \$390000.

6. (a) The matrix we want is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

which is the "within-two-minute" indicator matrix. More precisely, we construct the matrix A according to the distance matrix so that

$$A_{ij} = \begin{cases} 1 & \text{if cities } i \text{ and } j \text{ are within two minutes of drive} \\ 0 & \text{otherwise} \end{cases},\ i = 1, 2, ..., 8, j = 1, 2, ..., 8.$$

(b) Let

$$x_i = \begin{cases} 1 & \text{if we locate one ambulance in district } i \\ 0 & \text{otherwise} \end{cases},\ i = 1, 2, ..., 8, \text{ and}$$

$$y_i = \begin{cases} 1 & \text{if district } i \text{ is within two minutes of an ambulance} \\ 0 & \text{otherwise} \end{cases},\ i = 1, 2, ..., 8$$

[2] Note that if we plug in numbers, we will see that $K = 100 < \sum_{j=1}^{3} D_j = 190$. So choosing K is sufficient for this problem. Nevertheless, in general we should use $\min\left\{K, \sum_{j=1}^{3} D_j\right\}$.

be our decision variables. We also define $P = (40, 30, 35, 20, 15, 50, 45, 60)$ as the population vector. With matrix A defined in Part (a), the problem may now be formulated as

$$\begin{array}{rll} \max & \sum_{i=1}^{8} P_i y_i & \\ \text{s.t.} & \sum_{i=1}^{8} x_i = 2 & \\ & \sum_{j=1}^{8} A_{ij} x_j \geq y_i & \forall\, i = 1, ..., 8 \\ & x_i, y_i \in \{0, 1\} & \forall\, i = 1, ..., 8. \end{array}$$

The key idea here is the second constraint, $\sum_{j=1}^{8} A_{ij} x_j \geq y_i$ for all i. Consider district i. To have it within two minutes of an ambulance, we check all the eight districts (including itself) to see if there is any district that (1) is within two minutes of traveling and (2) has an ambulance. For district j, the first condition is $A_{ij} = 1$ and the second one is $x_j = 1$. For district j we need both of them, so we multiply them and use $A_{ij} x_j$. We need at least one district to have this, so we summate this for all j. If for district i the LHS is at least one, then we can choose $y_i = 1$ to maximize our objective value. The objective function gives us the number of people living within two minutes of an ambulance. Finally, the first constraint allows us to have only 2 ambulances.

(c) The MS Excel Solver program is contained in the MS Excel file "ORSp13_hw06_sol.xlsx" under the sheet "Problem 6". The optimal solution is to locate two ambulances at locations 4 and 7 so that districts 3 to 8 can all be covered. The number of people covered is 225 thousands.