Operations Research, Spring 2013
Suggested Solution for Homework 07

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1. The dual LP is

$$
\begin{array}{lrcrcrcr}
\min & y_1 & + & 3y_2 & + & 4y_3 & & \\
\text{s.t.} & -y_1 & + & y_2 & + & y_3 & \geq & 2 \\
 & y_1 & + & y_2 & - & 2y_3 & \geq & 1 \\
 & \multicolumn{7}{l}{y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0.}
\end{array}
$$

2. The dual program is

$$
\begin{array}{lrcrcrcrcr}
\min & 5y_1 & + & 7y_2 & + & 6y_3 & + & 4y_4 & & \\
\text{s.t.} & y_1 & + & 2y_2 & & & + & y_4 & \geq & 4 \\
 & y_1 & + & y_2 & + & 2y_3 & & & = & -1 \\
 & & & & & y_3 & + & y_4 & = & 2 \\
 & \multicolumn{9}{l}{y_1 \geq 0,\ y_2 \geq 0,\ y_3 \leq 0,\ y_4 \text{ urs.}}
\end{array}
$$

3. (a) The optimal simplex tableau is

$$
\begin{array}{ccccc|c}
3 & 0 & 0 & 4 & 1 & 300 \\
\hline
\frac{1}{2} & 0 & 1 & \frac{3}{2} & -\frac{1}{2} & x_3 = 25 \\
\frac{1}{2} & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & x_2 = 25
\end{array}
$$

(b) The primal optimal solution is $(x_1^*, x_2^*, x_3^*) = (0, 25, 25)$. As all the reduced costs are positive, the optimal solution is unique.

(c) With the optimal basis $B = \{x_3, x_2\}$, we have

$$
c_B A_B^{-1} = \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 7 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 & 1 \end{bmatrix}.
$$

Therefore, the shadow prices for constraints 1 and 2 are 4 and 1, respectively.

(d) The dual LP is

$$
\begin{array}{ll}
\min & 50y_1 + 100y_2 \\
\text{s.t.} & y_1 + 2y_2 \geq 3 \\
 & y_1 + 3y_2 \geq 7 \\
 & y_1 + y_2 \geq 5 \\
 & y_i \geq 0 \quad \forall\, i = 1, 2.
\end{array}
$$

Graphically we may show that the optimal solution lies at the intersection of $y_1 + 3y_2 = 7$ and $y_1 + y_2 = 5$, which is $(y_1^*, y_2^*) = (4, 1)$. This is exactly what we find in Part (c).

4. We run one iteration

$$
\begin{array}{cccc|c}
0 & -2 & 0 & 0 & 0 \\
\hline
1 & -1 & 1 & 0 & 4 \\
-1 & \boxed{1} & 0 & 1 & 1
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc|c}
-2 & 0 & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 1 & 5 \\
-1 & 1 & 0 & 1 & 1
\end{array}
$$

and find that the primal LP is unbounded, which implies that the dual LP is infeasible by strong duality.

5. (a) The dual problem is

$$\begin{array}{lrcrcrcl}
\min & 600y_1 & + & 400y_2 & + & 500y_3 & & \\
\text{s.t.} & 4y_1 & + & y_2 & + & 3y_3 & \geq & 6 \\
 & 9y_1 & + & y_2 & + & 4y_3 & \geq & 10 \\
 & 7y_1 & + & 3y_2 & + & 2y_3 & \geq & 9 \\
 & 10y_1 & + & 40y_2 & + & y_3 & \geq & 20 \\
 & & & & & y_i & \geq & 0 \quad \forall i = 1, 2, 3.
\end{array}$$

(b) The objective value associated with any dual optimal solution must be equal to $z^* = \frac{2800}{3}$ by strong duality.

(c) If we plug in the primal optimal solution into the primal LP, we will see that the third constraint is nonbinding at the optimal solution. Therefore, its shadow price must be zero. As this shadow price equals the third dual variable y_3, we may conclude that $y_3 = 0$.

(d) Suppose the first dual constraint is nonbinding, it follows that its shadow price, which equals the value of x_1 in the primal optimal solution, must be zero. As $x_1 > 0$ in the primal optimal solution, the first dual constraint must be binding at any dual optimal solution. Similarly, the fourth dual constraint must be binding at any dual optimal solution.