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Suggested Solution for Homework 08

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1. The first order derivatives are

$$\frac{\partial}{\partial x_1} f(x_1, x_2) = 2x_1 e^{x_2} \quad \text{and} \quad \frac{\partial}{\partial x_2} f(x_1, x_2) = x_1^2 e^{x_2}.$$

The second order derivatives are

$$\frac{\partial^2}{\partial x_1^2} f(x_1, x_2) = 2e^{x_2}, \quad \frac{\partial^2}{\partial x_2^2} f(x_1, x_2) = x_1^2 e^{x_2}, \text{ and}$$
$$\frac{\partial^2}{\partial x_1^2 x_2^2} f(x_1, x_2) = 2x_1 e^{x_2} = \frac{\partial^2}{\partial x_2^2 x_1^2} f(x_1, x_2).$$

Finally, the gradient is

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 e^{x_2} \\ x_1^2 e^{x_2} \end{bmatrix}.$$

2. With the decision variables S and F defined in the problem, the complete formulation is

$$\begin{aligned} \min \quad & 50S + 100F \\ \text{s.t.} \quad & 5\sqrt{S} + 17\sqrt{F} \geq 40 \\ & 20\sqrt{S} + 7\sqrt{F} \geq 60 \\ & S \geq 0, \ F \geq 0. \end{aligned}$$

The objective function minimizes the total cost. The first constraint ensures that the ads are seen by at least 40 million men. The second constraint ensures that the ads are seen by at least 60 million women. Lastly, we have nonnegativity constraints.

3. Let x_i be the production quantity at plant i, $i = 1, 2$. The complete formulation is

$$\begin{aligned} \max \quad & 10(x_1 + x_2) - 20\sqrt{x_1} - 40\sqrt[3]{x_2} \\ \text{s.t.} \quad & x_1 + x_2 \leq 120 \\ & x_1 \leq 70 \\ & x_2 \leq 70 \\ & x_1, \ x_2 \geq 0. \end{aligned}$$

The objective function maximizes the total profit. The first constraint limits the amount of total production to be no more than the demand size. the second and third constraints are the capacity constraints. Lastly, we have nonnegativity constraints.

4. To formulate this problem, we need to assign coordinates to the three cities. While there are infinitely many ways of doing so, below we choose the setting depicted in Figure 1. Note that the information that the distance between any two cities is a has been used in constructing this figure. Let (x, y) be the coordinates of the airport. Under this setting, the complete formulation is

$$\min \ \sqrt{x^2 + y^2} + \sqrt{(x-d)^2 + y^2} + \sqrt{\left(x - \frac{d}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{2}d\right)^2}$$

The objective function minimizes the total distance from the airport to the three cities. We do not need any constraint.

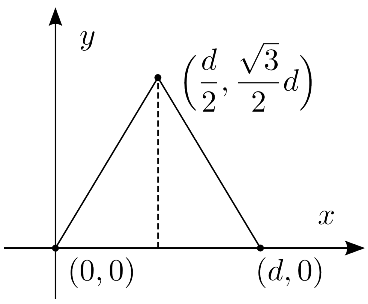

Figure 1: Coordinates of the three cities

5. First, note that the feasible region is convex (you may draw a graph to see this). Because the objective function is to minimize a linear function, which is convex, this is a convex program and thus a local optimum is a global optimum. Because we are also minimizing a concave function (because a linear function is also concave), the existence of an optimal solution implies the existence of an extreme point optimal solution.

6. First, note that the feasible region is convex (as all the constraints are linear). Now consider the objective function. Both $\sqrt{\cdot}$ and $\sqrt[3]{\cdot}$ are concave, and thus $-\sqrt{\cdot}$ and $-\sqrt[3]{\cdot}$ are convex. This implies that the objective function is to maximize a convex function. Therefore, the existence of an optimal solution implies the existence of an extreme point optimal solution. However, a local optimum may not be a global optimum.

7. (a) $f'(x) = 3x^2$ and $f''(x) = 6x$. As $6x \geq 0$ for all $x \geq 0$, f is convex.
   (b) $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$. As $\frac{2}{x^3} > 0$ for all $x > 0$, f is convex.
   (c) $f'(x) = 2\cos(2x)$ and $f''(x) = -4\sin(2x)$. As $-4\sin(2x) \leq 0$ for $x \in [\pi, \frac{3}{2}\pi]$ but $-4\sin(2x) \geq 0$ for $x \in [\frac{3}{2}\pi, 2\pi]$, this function is neither convex nor concave.
   (d) $f'(x) = ax^{a-1}$ and $f''(x) = a(a-1)x^{a-2}$. As $a(a-1)x^{a-2} \leq 0$ for all $x \geq 0$ for any $a \in (0, 1)$, f is concave.
   (e) $f'(x) = (\ln 2)(2^x)$ and $f''(x) = (\ln 2)^2(2^x)$. As $(\ln 2)^2(2^x) > 0$ for all $x \in \mathbb{R}$, f is convex.

8. For the original problem without tax, the formulation is

$$\begin{aligned} \max \quad & \pi_1(q) = q(100 - 4q) - 50 - 5q \\ \text{s.t.} \quad & q \geq 0 \\ & 100 - 4q \geq 0. \end{aligned}$$

The objective function is to maximize the sales revenue $q(100 - 4q)$ minus the fixed cost 50 and the variable cost $5q$. The two constraints ensure that the production quantity and the market price are both nonnegative. To solve this program, note that

$$\pi_1'(q) = 100 - 4q - 4q - 5 = -8q + 95 \quad \text{and} \quad \pi_1''(q) = -8 < 0.$$

so the objective function is maximizing a concave function and thus a feasible point satisfying the FOC is optimal. As $q^* = \frac{95}{8}$ is such a point, it is the optimal production quantity.

When there is the sales tax, the new formulation is

$$\begin{aligned} \max \quad & \pi_2(q) = q(100 - 4q) - 50 - 5q - 2q \\ \text{s.t.} \quad & q \geq 0, \\ & 100 - 4q \geq 0. \end{aligned}$$

The only difference is that in the objective function, there is an additional term representing the sales tax. As

$$\pi_2'(q) = -8q - 93 \quad \text{and} \quad \pi_2''(q) = -8 < 0,$$

the new program is still convex and we again look for the point satisfying the FOC, which is $q^{**} = \frac{93}{8}$. As it is feasible, it is optimal. Note that $q^{**} < q^*$: As the net sales revenue decreases due to the existence of the sales tax, the optimal production quantity decreases.

9. (a) Figure 2 depicts the feasible region and one isoprofit curve (the dotted curve). For solving this problem, we look for the feasible point that is closest to the point $(3, 1)$ (which is the center of all isoprofit "circles"). Such a point $(x_1, x_2) = (2, 2)$ is the optimal solution to this problem.

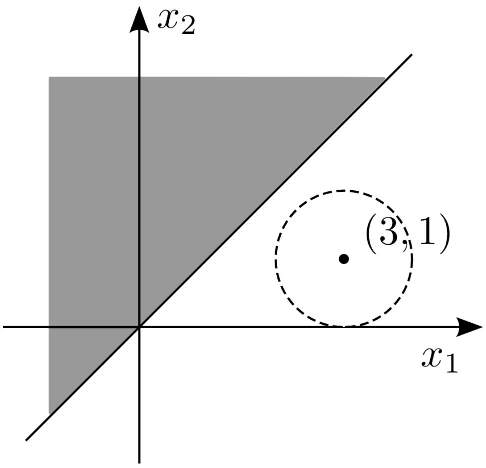

Figure 2: Graphical solution for Problem 9

(b) The Lagrangian relaxation is

$$L(\lambda) = \min_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} (x_1 - 3)^2 + (x_2 - 1)^2 + \lambda(-x_1 + x_2)$$

for some $\lambda \leq 0$. To verify that λ should be nonpositive, recall that we want to give penalty to a point that violates the constraint. When the constraint is violated, we have $x_1 - x_2 > 0$ and thus $-x_1 + x_2 < 0$. To make the solution worse for this minimization objective function, we should make the objective larger by making the last term nonnegative. This requires λ to be nonpositive.

(c) To solve the Lagrangian relaxation, note that the objective function can be decomposed into two separated functions, one for x_1 and one for x_2. We may also verify that both functions are convex and thus the FOC is sufficient for a global optimum. Let x_1^* and x_2^* be the points satisfying the FOC, we have

$$2(x_1^* - 3) - \lambda = 0 \quad \text{and} \quad 2(x_2^* - 1) + \lambda = 0,$$

which imply $x_1^* = \frac{6+\lambda}{2}$ and $x_2^* = \frac{2-\lambda}{2}$. We may then plug in x_1^* and x_2^* into the objective function and obtain $L(\lambda) = \frac{\lambda^2}{4} + \frac{\lambda^2}{4} + \lambda(-2 - \lambda) = -\frac{\lambda^2}{2} - 2\lambda$.

(d) To see that $L(\lambda)$ is concave in λ, note that $L'(\lambda) = -\lambda - 2$ and $L''(\lambda) = -1 < 0$.

(e) The Lagrangian dual program is

$$\begin{aligned} \max \quad & -\frac{\lambda^2}{2} - 2\lambda \\ \text{s.t.} \quad & \lambda \leq 0. \end{aligned}$$

By ignoring the constraint and apply the FOC, we find that the optimal dual solution is $\lambda^* = -2$. By plugging in λ^* into $x_1^* = \frac{6+\lambda}{2}$ and $x_2^* = \frac{2-\lambda}{2}$, we obtain $x_1^* = 2$ and $x_2^* = 2$, which exactly form the primal optimal solution.