Operations Research, Spring 2013
Homework 10

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1. (10 points; 5 points each) Consider the following prisoners' dilemma for two players

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

we discussed in class. Suppose that before they broke the window, the two players have signed a contract for the following agreements: "If A confesses while B denies, after B finishes his time in prison, A must serve as B's slave for b months." With this agreement, the payoff matrix becomes

	Denial	Confession
Denial	$-1, -1$	$-9 + b, -b$
Confession	$-b, -9 + b$	$-6, -6$

   (a) Suppose $b = 2$. Find all the Nash equilibria, if any. For each Nash equilibrium you find, you need to explain why it is a Nash equilibrium.

   (b) Suppose $b = 4$. Find all the Nash equilibria, if any. For each Nash equilibrium you find, you need to explain why it is a Nash equilibrium.

   Note. This problem shows that social efficiency can be achieved with appropriate agreements, given that the agreement is credible.

2. (Adopted from Figure 37.1 of Osborne 2004; 20 points, 5 points each) Consider the following two-player static game

	L	C	R
T	$1, 2$	$2, 1$	$1, 0$
M	$2, 1$	$0, 1$	$0, 0$
B	$0, 1$	$0, 0$	$1, 2$

   Let player 1 be choosing T, M, or B and player 2 be choosing L, C, or R.

   (a) Write down player 1's best response function.

   (b) Write down player 2's best response function.

   (c) Does any player have any strategy that is strictly dominated by his another strategy?

   (d) Using your answers in Parts (a) and (b), find all the Nash equilibria, if any. Explain your answer with the two best response functions.

   Hint. When there is a tie, the best response function may return a *set* rather than a *single value*.

3. (20 points; 5 points each) In a small town, there are n persons that may contribute to the construction of a library. Each of them may contribute either \$1 or \$0 and the library will be built once there are at least \$$k$ contributed, where $k < n$. However, if fewer than k people contribute, the library will not be built and the money will not be refunded. Each person will feel like earning \$3 if the library is built and \$0 otherwise. Each person's utility function is the virtual earning from the library (if any) minus the amount she/he contributes.

(a) Suppose $n = 2$ and $k = 1$. Construct the payoff matrix for the two players. Find all the Nash equilibria, if any. Explain your answers.

(b) Suppose $n = 3$ and $k = 2$. Construct the "three-dimentional" payoff matrix

	0		1	
	0	1	0	1
0	$0, 0, 0$	$0, -1, 0$	$0, 0, -1$	$3, 2, 2$
1	$-1, 0, 0$	$2, 2, 3$	$2, 3, 2$	$2, 2, 2$

by filling numbers into the eight cells. For this matrix, cell 1 is for the case that no player contributes, cell 2 is for the case that only player 2 contributes, cell 3 is for the case that only player 3 contributes, cell 4 is for the case that players 2 and 3 contributes, etc. For each of the eight cells, you need to put three numbers, where the ith number is player i's payoff under that action profile.

(c) Following from Part (b), find all the Nash equilibria, if any. Explain your answers.

(d) For general n and k, let x be the number of persons contributing under a Nash equilibrium. What are the possible values of x?

4. (Modified from Problem 1.4 in Gibbons 1992; 30 points, 10 points each) Suppose there are n firms engaged in a Cournot competition. Let q_i denote the quantity produced by firm i, $i = 1, ..., n$, and let $Q = q_1 + \cdots + q_n$ denote the aggregate quantity on the market. Let P denote the market-clearing price and assume that inverse demand is given by $P(Q) = a - Q$ if $Q \leq a$ or 0 otherwise. Assume that the unit production cost $c < a$. These firms choose their quantities simultaneously.

(a) Suppose $n = 3$, find the unique Nash equilibrium. In equilibrium, what are the aggregate demand and price? Show that the aggregate demand becomes higher and the price becomes lower when the number of firms changes from two to three.

(b) For general n, find the unique Nash equilibrium. What are the equilibrium aggregate demand and price?

(c) When n approaches infinity, what values do the equilibrium aggregate demand and price approach? What are the implications of these results?

5. (Modified from Problem 1.6 in Gibbons 1992; 20 points) Suppose in a Cournot competition with two firms, the market-clearing price is still $P(Q) = a - Q$, where $Q = q_1 + q_2$ and q_i is the production quantity of firm 1. However, firms now have asymmetric unit production costs: c_1 for firm 1 and c_2 for firm 2. Without loss of generality, assume $c_1 < c_2$.

(a) (5 points) Suppose $0 < c_i < \frac{a}{2}$ for $i = 1, 2$. What is the Nash equilibrium? Which firm produces more? Does that make sense?

(b) (10 points) Suppose $c_1 = 2$, $c_2 = 8$, and $a = 10$. Write down the two best response functions and then depict them for $q_1 \in [0, 8]$ and $q_2 \in [0, 8]$.

Hint. For a wide range of q_1, the optimal q_2 is the same.

(c) (5 points) Suppose $c_1 < c_2 < a$ and $2c_2 > a + c_1$. What is the Nash equilibrium?