IM2010: Operations Research
The Two-Phase Implementation
(Chapter 4)

Ling-Chieh Kung

Department of Information Management
National Taiwan University

April 8, 2013

Feasibility of a linear program

- When a linear program

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0, \end{aligned}$$

  satisfies $b \geq 0$, finding a basic feasible solution for its standard form is trivial.
  - We may form a feasible basis with all the slack variables.
- However, if there are some **equality** or **no-less-than** constraints, finding a feasible basis can be hard.

Feasibility of a linear program

- For example, given a linear program

$$
\begin{array}{llllllllll}
\min & x_1 \\
\text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & \geq & 10 \\
& 3x_1 & + & 2x_2 & + & 9x_3 & - & x_4 & = & 10 \\
& x_1 & - & 8x_2 & + & 2x_3 & - & 6x_4 & \leq & 10 \\
& \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4}
\end{array}
$$

  and its standard form

$$
\begin{array}{llllllllllll}
\min & x_1 \\
\text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & - & x_5 & & = & 10 \\
& 3x_1 & + & 2x_2 & + & 9x_3 & - & x_4 & & & & = & 10 \\
& x_1 & - & 8x_2 & + & 2x_3 & - & 6x_4 & & & +x_6 & = & 10 \\
& \multicolumn{11}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6,}
\end{array}
$$

  it is nontrivial to find a feasible basis (if there is one).

Feasibility of a linear program

- As the simplex method requires an **initial basic feasible solution** to start from, we must have an efficient way to find one if there is at least one.
- To find an initial basic feasible solution, there are at least two ways to implement the simplex method:
  - The two-phase method.
  - The big-M method.
- Here we introduce the **two-phase implementation**.
  - The big-M method is conceptually identical.
  - However, the two-phase method is typically more efficient.

Road map

- **The idea**.
- Feasible examples.
- Infeasible examples.

The two-phase implementation

- Consider a linear program and its standard form

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & A_1 x \le b_1 \\ & A_2 x = b_2 \\ & A_3 x \ge b_3 \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & A_1 x + y = b_1 \\ & A_2 x = b_2 \\ & A_3 x - z = b_3 \\ & x, y, z \ge 0, \end{aligned}$$

  - y and z are slack variables.
  - As we mentioned above, it is generally hard to find a basic feasible solution for the standard form.
  - Let's construct a **Phase-I linear program** that has a **trivial** basic feasible solution.

Artificial variables

- For each equality or no-less-than constraint, we add a nonnegative **artificial variable**.

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & A_1 x \leq b_1 \\ & A_2 x = b_2 \\ & A_3 x \geq b_3 \\ & x \geq 0 \end{aligned} \qquad\qquad \begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & A_1 x + y = b_1 \\ & A_2 x + a_2 = b_2 \\ & A_3 x - z + a_3 = b_3 \\ & x, y, z, a_2, a_3 \geq 0, \end{aligned}$$

  - a_2 and a_3 are artificial variables.
  - Unlike slack variables which have **physical meanings** (e.g., the gap between the two sides), artificial variables are added purely for checking feasibility.

Artificial variables

- For example, if the original linear program is

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & - & x_2 & \leq & 10 \\ & 2x_1 & + & x_2 & \geq & 6 \\ & x_1 & + & 2x_2 & = & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2,} \end{array}$$

  then two artificial variables x_5 and x_6 should be added:

$$\begin{array}{rrcrcrcrcrcrcl} \max & x_1 & + & x_2 & & & & & & & & & & \\ \text{s.t.} & x_1 & - & x_2 & + & x_3 & & & & & & & = & 10 \\ & 2x_1 & + & x_2 & & & - & x_4 & + & x_5 & & & = & 6 \\ & x_1 & + & 2x_2 & & & & & & & + & x_6 & = & 6 \\ & \multicolumn{13}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6.} \end{array}$$

A trivial basic feasible solution

- Why do we add artificial variables?
- Because we may find an **identity** matrix in the constraint matrix!
- For each constraint, we need a variable that has **coefficient 1** and appears **only** at this constraint.
  - For a no-greater-than constraint, the slack variable works.
  - For an equality or no-less-than constraint, the artificial variable works.
- Forming a basis by including these variables definitely results in a basic feasible solution.
- In the previous example, $B = \{x_3, x_5, x_6\}$ is a feasible basis.

Artificial variables are not real

- However, artificial variables are not real.
- The basic feasible solution with **positive** artificial variables does **not** correspond to a basic feasible solution to the original standard form linear program.
  - The basic feasible solution of $B = \{x_3, x_5, x_6\}$ is $(0, 0, 10, 0, 6, 6)$.
  - If we remove the artificial variables, the corresponding solution is $(0, 0, 10, 0)$, which is infeasible to

$$
\begin{array}{rrcrcrcrcl}
\max & x_1 & + & x_2 & & & & & & \\
\text{s.t.} & x_1 & - & x_2 & + & x_3 & & & = & 10 \\
 & 2x_1 & + & x_2 & & & - & x_4 & = & 6 \\
 & x_1 & + & 2x_2 & & & & & = & 6 \\
 & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6.}
\end{array}
$$

Removing artificial variables

- So the question is: For the linear program with artificial variables, is there **any** basic feasible solution whose artificial variables are all **zero**?
  - If the answer is yes, the original program is **feasible**. Removing all artificial variables results in a basic feasible solution for the original standard form.
  - If no, the original program is **infeasible**.
- Interestingly, we may use the simplex method to answer this!
- All we need is to "design" a new objective function that drives the artificial variables to zero.

Phase-I linear programs

- In the Phase-I linear program, we will ignore the original objective function and **minimize the sum of artificial variables** instead.

$$
\begin{array}{llllllllllll}
\min & & & & & & & x_5 & + & x_6 & & \\
\text{s.t.} & x_1 & - & x_2 & + & x_3 & & & & & = & 10 \\
& 2x_1 & + & x_2 & & & - & x_4 & + & x_5 & & = & 6 \\
& x_1 & + & 2x_2 & & & & & & + & x_6 & = & 6 \\
& x_i \geq 0 & & \forall i = 1, ..., 6. & & & & & & & &
\end{array}
$$

- Once we use the simplex method to solve this Phase-I program:
  - If there is a basic feasible solution with $x_5 = x_6 = 0$, we will definitely find it.
  - If there is none, we will end up with an "optimal" solution which still have at least one positive artificial variable.
  - We may also maximize the negative sum of artificial variables.

The two-phase implementation

- Given a linear program, we first construct its Phase-I program:
  - Add artificial variables for equality and no-less-than constraints.
  - Minimize the sum of artificial variables.
- We then use the simplex method to solve the Phase-I program:
  - We start from the trivial basic feasible solution.
  - If we cannot remove all artificial variables, the original program is infeasible.
  - Otherwise, we obtain a feasible basis for the original program. We put the original objective function back and start **Phase II** to solve the original program.

Road map

- The idea.
- **Feasible examples**.
- Infeasible examples.

Example 1: Phase-I program

- Consider a linear program

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \geq & 6 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2.} \end{array}$$

- Its Phase-I linear program is

$$\begin{array}{rrcrcrcrcrcl} \max & & & & & & - & x_4 & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & & & = & 6 \\ & x_1 & + & 2x_2 & & & & & + & x_5 & = & 6 \\ & \multicolumn{11}{l}{x_i \geq 0 \quad \forall i = 1, ..., 5.} \end{array}$$

Example 1: preparing the initial tableau

- Let's try to solve the Phase-I program. First let's prepare the initial tableau:

0	0	0	1	0	0
2	1	-1	1	0	$x_4 = 6$
1	2	0	0	1	$x_5 = 6$

- Is this a valid tableau? No!
  - For all basic columns (in this case, columns 4 and 5), the 0th row should contain 0.
  - So we need to first **fix the 0th row** through elementary row operations.

Example 1: preparing the initial tableau

- Let's fix the 0th row by adding the negation of the 1st row into the 0th row.

0	0	0	1	0	0
2	1	−1	1	0	$x_4 = 6$
1	2	0	0	1	$x_5 = 6$

→

−2	−1	1	0	0	−6
2	1	−1	1	0	$x_4 = 6$
1	2	0	0	1	$x_5 = 6$

- Now we have a valid initial tableau to start from!
- The current basic feasible solution is $x^0 = (0, 0, 0, 6, 6)$, which corresponds to an **infeasible** solution to the original program.
  - We know this because there are positive artificial variables.

Example 1: solving the Phase-I program

- Solving the Phase-I program takes only one iteration:

$$
\begin{array}{ccccc|c}
-2 & -1 & 1 & 0 & 0 & -6 \\
\hline
\boxed{2} & 1 & -1 & 1 & 0 & x_4 = 6 \\
1 & 2 & 0 & 0 & 1 & x_5 = 6
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc|c}
0 & 0 & 0 & 1 & 0 \\
\hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \frac{3}{2} & \frac{1}{2} & 1 & x_5 = 3
\end{array}
$$

  - Whenever an artificial variable leaves the basis, we will not need to enter it again. Therefore, we can remove the corresponding column to save some time.
- Because we successfully remove all the artificial variable, the original program is feasible.
- The initial basis for the original program is $\{x_1, x_5\}$.

Example 1: solving the Phase-II program

- Now we may solve the Phase-II program (which is just the original program).
- First, let's put the original objective function back:

$$
\begin{array}{cccc|c}
-1 & -1 & 0 & 0 & 0 \\
\hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \frac{3}{2} & \frac{1}{2} & 1 & x_5 = 3
\end{array}
$$

  - From the last tableau to this one, we only modify the 0th row. All other rows remain unchanged.
- Is this a valid tableau? No!
  - Column 1, which is basic, contains a nonzero number in the 0th row. It must be fixed to 0.
- Before we run iterations, let's fix the 0th row again.

Example 1: solving the Phase-II program

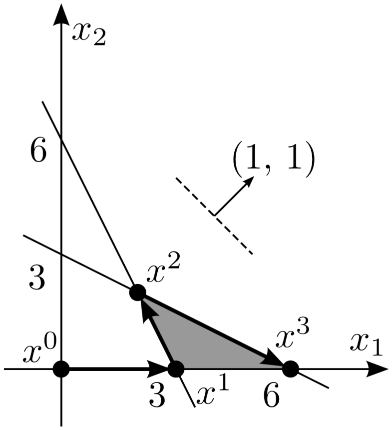
- Let's fix the 0th row and then run two iterations:

$$
\begin{array}{cccc|c}
-1 & -1 & 0 & 0 & 0 \\
\hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \frac{3}{2} & \frac{1}{2} & 1 & x_5 = 3
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 3 \\
\hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \boxed{\frac{3}{2}} & \frac{1}{2} & 1 & x_5 = 3
\end{array}
$$

$$
\rightarrow
\begin{array}{cccc|c}
0 & 0 & -\frac{1}{3} & \frac{1}{3} & 4 \\
\hline
1 & 0 & -\frac{2}{3} & -\frac{1}{3} & x_1 = 2 \\
0 & 1 & \boxed{\frac{1}{3}} & \frac{2}{3} & x_2 = 2
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & 1 & 0 & 1 & 6 \\
\hline
1 & 2 & 0 & 1 & x_1 = 6 \\
0 & 3 & 1 & 2 & x_3 = 6
\end{array}
$$

- The optimal basic feasible solution is $(6, 0, 6, 0)$.

Example 1: visualization



- x^0 is infeasible ($x_4 > 0$).
- x^1 is the initial bfs (as a result of Phase I).
- x^3 is the optimal bfs (as a result of Phase II).

Example 2: Phase-I program

- Consider a linear program

$$\begin{array}{rrcrcr} \max & x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \geq & 6 \\ & x_1 & + & 2x_2 & = & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2} \end{array}$$

  and its Phase-I program

$$\begin{array}{rrcrcrcrcrcr} \max & & & & & & - & x_4 & - & x_5 & & \\ \text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & & & = & 6 \\ & x_1 & + & 2x_2 & & & & & + & x_5 & = & 6 \\ & \multicolumn{11}{l}{x_i \geq 0 \quad \forall i = 1, ..., 5.} \end{array}$$

Example 2: solving the Phase-I program

- We first fix the 0th row and then run two iterations to remove all the artificial variables:

$$
\begin{array}{ccccc|c}
0 & 0 & 0 & 1 & 1 & 0 \\ \hline
2 & 1 & -1 & 1 & 0 & x_4 = 6 \\
1 & 2 & 0 & 0 & 1 & x_5 = 6
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
-3 & -3 & 1 & 0 & 0 & -12 \\ \hline
\boxed{2} & 1 & -1 & 1 & 0 & x_4 = 6 \\
1 & 2 & 0 & 0 & 1 & x_5 = 6
\end{array}
$$

$x^0 = (0, 0, 0, \underline{6}, \underline{6})$ is infeasible

$$
\rightarrow \quad
\begin{array}{cccc|c}
0 & -\frac{3}{2} & -\frac{1}{2} & 0 & -3 \\ \hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \boxed{\frac{3}{2}} & \frac{1}{2} & 1 & x_5 = 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccc|c}
0 & 0 & 0 & 0 \\ \hline
1 & 0 & -\frac{2}{3} & x_1 = 2 \\
0 & 1 & \frac{1}{3} & x_2 = 2
\end{array}
$$

$x^1 = (3, 0, 0, \underline{0}, \underline{3})$ is infeasible

$x^2 = (2, 2, 0, \underline{0}, \underline{0})$ is feasible

Example 2: solving the Phase-II program

- With the initial basis $\{x_1, x_2\}$, we then solve the Phase-II program in one iteration (do not forget to fix the 0th row):

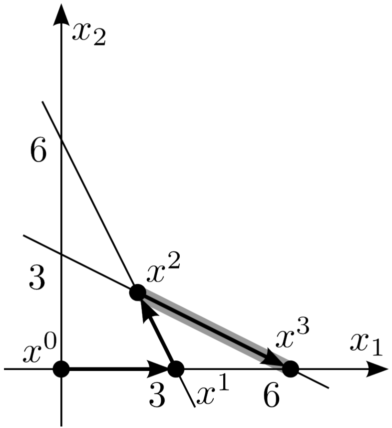
$$\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ \hline 1 & 0 & -\frac{2}{3} & x_1 = 2 \\ 0 & 1 & \frac{1}{3} & x_2 = 2 \end{array} \rightarrow \begin{array}{ccc|c} 0 & 0 & -\frac{1}{3} & 4 \\ \hline 1 & 0 & -\frac{2}{3} & x_1 = 2 \\ 0 & 1 & \boxed{\frac{1}{3}} & x_2 = 2 \end{array}$$

$x^2 = (2, 2, 0)$ is not optimal

$$\rightarrow \begin{array}{ccc|c} 0 & 1 & 0 & 6 \\ \hline 1 & 2 & 0 & x_1 = 6 \\ 0 & 3 & 1 & x_3 = 6 \end{array}$$

$x^3 = (6, 0, 6)$ is optimal

Example 2: visualization



- x^0 and x^1 are infeasible.
- x^2 is the initial bfs (as a result of Phase I).
- x^3 is the optimal bfs (as a result of Phase II).

Road map

- The idea.
- Feasible examples.
- **Infeasible examples**.

Example: Phase-I program

- Consider the linear program

$$\begin{array}{rrcrcl} \max & x_1 & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & x_2 & = & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2} \end{array}$$

  and its Phase-I program

$$\begin{array}{rrcrcrcrcl} \min & & & & & & & x_4 & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & x_1 & + & x_2 & & & + & x_4 & = & 6 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$

Example: solving the Phase-I program

- After fixing the 0th row, we run two iterations...

$$\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 0 \\ \hline 2 & 1 & 1 & 0 & x_3 = 4 \\ 1 & 1 & 0 & 1 & x_4 = 6 \end{array} \rightarrow \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 6 \\ \hline \boxed{2} & 1 & 1 & 0 & x_3 = 4 \\ 1 & 1 & 0 & 1 & x_4 = 6 \end{array}$$

$x^0 = (0, 0, 4, \underline{6})$ is infeasible

$$\rightarrow \begin{array}{cccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 4 \\ \hline 1 & \boxed{\frac{1}{2}} & \frac{1}{2} & 0 & x_1 = 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & x_4 = 4 \end{array} \rightarrow \begin{array}{cccc|c} -1 & 0 & -1 & 0 & 2 \\ \hline 2 & 1 & 1 & 0 & x_2 = 4 \\ -1 & 0 & -1 & 1 & x_4 = 2 \end{array}$$

$x^1 = (0, 2, 0, \underline{4})$ is infeasible

$x^2 = (0, 4, 0, \underline{2})$ is infeasible

Example: solving the Phase-I program

- The final tableau

$$\begin{array}{cccc|c} -1 & 0 & -1 & 0 & 2 \\ \hline 2 & 1 & 1 & 0 & x_2 = 4 \\ -1 & 0 & -1 & 1 & x_4 = 2 \end{array}$$

  is optimal (for the Phase-I program).
- However, in the optimal solution $(0, 4, 0, 2)$, the artificial variable x_4 is still in the basis (and positive).
- Therefore, we conclude that the original program is infeasible.

Example: visualization

- The feasible region of the original program is empty.