IM2010: Operations Research
Duality
(Chapter 6)

Ling-Chieh Kung

Department of Information Management
National Taiwan University

April 25, 2013

Road map

- **Primal-dual pairs**.
- Properties of dual programs.
- Shadow prices.

Upper bounds of a maximization LP

- Consider the following LP

$$
\begin{array}{rrcrcrcc}
z^* = \max & 4x_1 & + & 5x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.}
\end{array}
$$

- Is there any way to find an **upper bound** of z^* without solving this LP?

Upper bounds of a maximization LP

- How about this: Multiplying the first constraint by 2, multiply the second constraint by 1, and then add them together.
- This creates a redundant constraint

$$\begin{aligned} & 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) \leq 2 \times 6 + 4 \\ \Rightarrow\ & 4x_1 + 5x_2 + 8x_3 \leq 16. \end{aligned}$$

- If we add this constraint into the LP:

$$\begin{array}{rrcrcrcr} z^* = \max & 4x_1 & + & 5x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & 4x_1 & + & 5x_2 & + & 8x_3 & \leq & 16 \\ & & & \multicolumn{5}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.} \end{array}$$

- So an upper bound of z^* is 16.

Upper bounds of a maximization LP

- How to find an upper bound of z^* for the following LP?

$$\begin{array}{rrcrcrcl} z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.} \end{array}$$

- Let's play the same trick. We will see 16 is also an upper bound:

$$\begin{aligned} & 2 \times 6 + 4 = 16 \\ \geq\ & 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) \\ =\ & 4x_1 + 5x_2 + 8x_3 \\ \geq\ & 3x_1 + 4x_2 + 8x_3. \qquad (\text{because } x_1 \geq 0,\ x_2 \geq 0) \end{aligned}$$

- How to find a **lower** upper bound?

Better upper bounds?

$$
\begin{array}{rrcrcrcl}
z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.}
\end{array}
$$

- Suppose we are going to **linearly combine** the two constraints with coefficients y_1 and y_2, respectively.
- Suppose $y_1 \geq 0$ and $y_2 \geq 0$ (why do we need this?). If

$$3 \leq y_1 + 2y_2, \quad 4 \leq 2y_1 + y_2, \quad \text{and } 8 \leq 3y_1 + 2y_2,$$

then we have

$$3x_1 + 4x_2 + 8x_3 \leq 6y_1 + 4y_2,$$

which means $6y_1 + 4y_2$ is an upper bound of z^*.

Looking for the lowest upper bound

- To look for the **lowest** upper bound, we solve **another LP**!

$$\begin{array}{rrcrcl} \min & 6y_1 & + & 4y_2 & & \\ \text{s.t.} & y_1 & + & 2y_2 & \geq & 3 \\ & 2y_1 & + & y_2 & \geq & 4 \\ & 3y_1 & + & 2y_2 & \geq & 8 \\ & & & \multicolumn{3}{l}{y_1 \geq 0,\ y_2 \geq 0.} \end{array}$$

- We call the original LP the **primal** LP.
- We define this LP, which generates the lowest upper bound of the primal LP, as its **dual** LP.
- This idea applies to any LP. Let's see more examples.

Nonpositive or free variables

- Suppose variables are not all nonnegative:

$$\begin{array}{rrcrcrcl} z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \leq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}} \end{array}$$

- With $y_1 \geq 0$ and $y_2 \geq 0$ as the coefficients, if we want

$$\begin{aligned} 3x_1 + 4x_2 + 8x_3 &\leq y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \\ &= (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2), \end{aligned}$$

  what are the new conditions we need?

Nonpositive or free variables

- To have

$$\begin{array}{rcccc} & 3x_1 & + & 4x_2 & + & 8x_3 \\ \leq & (y_1 + 2y_2)x_1 & + & (2y_1 + y_2)x_2 & + & (3y_1 + 2y_2)x_3, \end{array}$$

now we need

$$\begin{array}{rcrcl l} y_1 & + & 2y_2 & \geq & 3 & \text{because } x_1 \geq 0, \\ 2y_1 & + & y_2 & \leq & 4 & \text{because } x_2 \leq 0, \text{ and} \\ 3y_1 & + & 2y_2 & = & 8 & \text{because } x_3 \text{ is free.} \end{array}$$

- So the dual LP is

$$\begin{array}{rrcrcl} \min & 6y_1 & + & 4y_2 & & \\ \text{s.t.} & y_1 & + & 2y_2 & \geq & 3 \\ & 2y_1 & + & y_2 & \leq & 4 \\ & 3y_1 & + & 2y_2 & = & 8 \\ & \multicolumn{5}{c}{y_1 \geq 0, \; y_2 \geq 0.} \end{array}$$

No-less-than and equality constraints

- Suppose constraints are not all no-greater-than:

$$\begin{array}{rrcrcrcc}
z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & = & 4 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}}
\end{array}$$

- As we need an upper bound of z^*, we need to combine the two constraints so that the RHS is no less than the LHS. How to choose the sign of y_1 and y_2 to do that?
  - That is, how to get this **no-greater-than** inequality

$$y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \leq 6y_1 + 4y_2?$$

No-less-than and equality constraints

- To obtain

$$y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \leq 6y_1 + 4y_2,$$

  we only need $y_1 \leq 0$. y_2 can be of any sign (i.e., free).
- We still need

$$3 \leq y_1 + 2y_2, \quad 4 \geq 2y_1 + y_2, \quad \text{and } 8 = 3y_1 + 2y_2$$

  to obtain

$$3x_1 + 4x_2 + 8x_3 \leq y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3)$$

No-less-than and equality constraints

$$
\begin{array}{rrcrcrcl}
z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & = & 4 \\
 & \multicolumn{7}{l}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}}
\end{array}
$$

- As a summary, an upper bound is obtained as follows:

$$
\begin{aligned}
6y_1 + 4y_2 &\geq y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \\
&= (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2) \\
&\geq 3x_1 + 4x_2 + 8x_3,
\end{aligned}
$$

  where the first inequality requires

$$y_1 \leq 0, y_2 \text{ free.}$$

  and the second inequality requires

$$3 \leq y_1 + 2y_2, \quad 4 \geq 2y_1 + y_2, \quad \text{and } 8 = 3y_1 + 2y_2.$$

No-less-than and equality constraints

- So for the primal LP

$$
\begin{array}{rrcrcrcl}
z^* = \max & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\
& 2x_1 & + & x_2 & + & 2x_3 & = & 4 \\
& \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.},}
\end{array}
$$

the dual LP is

$$
\begin{array}{rrcrcl}
\min & 6y_1 & + & 4y_2 & & \\
\text{s.t.} & y_1 & + & 2y_2 & \geq & 3 \\
& 2y_1 & + & y_2 & \leq & 4 \\
& 3y_1 & + & 2y_2 & = & 8 \\
& \multicolumn{5}{c}{y_1 \leq 0,\ y_2 = 0.}
\end{array}
$$

The general rule

- In general, if the primal LP is

$$\begin{array}{rrcrcrcc}
\max & c_1x_1 & + & c_2x_2 & + & c_3x_3 & & \\
\text{s.t.} & A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & \geq & b_1 \\
 & A_{21}x_1 & + & A_{22}x_2 & + & A_{23}x_3 & \leq & b_2 \\
 & A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & = & b_3 \\
 & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.},}
\end{array}$$

  its dual LP is

$$\begin{array}{rrcrcrcc}
\min & b_1y_1 & + & b_2y_2 & + & b_3y_3 & & \\
\text{s.t.} & A_{11}y_1 & + & A_{21}y_2 & + & A_{31}y_3 & \geq & 0 \\
 & A_{12}y_1 & + & A_{22}y_2 & + & A_{32}y_3 & \leq & 0 \\
 & A_{13}y_1 & + & A_{23}y_2 & + & A_{33}y_3 & = & 0 \\
 & \multicolumn{7}{c}{y_1 \leq 0,\ y_2 \geq 0,\ y_3 \text{ urs.}}
\end{array}$$

The dual LP for a minimization primal LP

- For a minimization primal LP, its dual LP is to find the **greatest lower bound**.
- Suppose the primal LP is

$$\begin{array}{rrcrcrcl} \min & 3x_1 & + & 4x_2 & + & 8x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\ & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\ & \multicolumn{7}{c}{x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.}} \end{array}$$

What conditions do we need to obtain the following lower bound?

$$\begin{aligned} 6y_1 + 4y_2 &\leq y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \\ &= (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2) \\ &\leq 3x_1 + 4x_2 + 8x_3, \end{aligned}$$

The dual LP for a minimization primal LP

- For the primal LP

$$
\begin{array}{rrcrcrcc}
\min & 3x_1 & + & 4x_2 & + & 8x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & \geq & 6 \\
 & 2x_1 & + & x_2 & + & 2x_3 & \leq & 4 \\
\end{array}
$$
$$x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ urs.},$$

  the dual LP is

$$
\begin{array}{rrcrcc}
\max & 6y_1 & + & 4y_2 & & \\
\text{s.t.} & y_1 & + & 2y_2 & \leq & 3 \\
 & 2y_1 & + & y_2 & \geq & 4 \\
 & 3y_1 & + & 2y_2 & = & 8 \\
\end{array}
$$
$$y_1 \geq 0,\ y_2 \leq 0.$$

The general rule

- The general rule for finding the dual LP is summarized below:

Obj. function	max	min	Obj. function
Constraint	$\leq$ $\geq$ $=$	≥ 0 ≤ 0 urs.	Variable
Variable	≥ 0 ≤ 0 urs.	$\geq$ $\leq$ $=$	Constraint

  - If the primal LP is a maximization problem, do it from left to right.
  - If the primal LP is a minimization problem, do it from right to left.

Examples of primal-dual pairs

- Example 1:

$$\begin{array}{rrcrcr} \min & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & 4x_1 & + & x_2 & \leq & 9 \\ & x_1 & & & \geq & 6 \\ & 2x_1 & - & x_2 & \geq & 8 \\ & \multicolumn{5}{l}{x_1 \leq 0,\ x_2 \text{ urs.}} \end{array} \Leftrightarrow \begin{array}{rrcrcrcr} \max & 9y_1 & + & 6y_2 & + & 8y_3 & & \\ \text{s.t.} & 4y_1 & + & y_2 & + & 2y_3 & \geq & 2 \\ & y_1 & & & - & y_3 & = & 3 \\ & \multicolumn{7}{l}{y_1 \leq 0,\ y_2 \geq 0,\ y_3 \geq 0.} \end{array}$$

- Example 2:

$$\begin{array}{rrcrcr} \max & 3x_1 & - & x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & = & 6 \\ & 3x_1 & + & 3x_2 & \leq & -4 \\ & \multicolumn{5}{l}{x_1 \text{ urs., } x_2 \geq 0.} \end{array} \Leftrightarrow \begin{array}{rrcrcr} \min & 6y_1 & - & 4y_2 & & \\ \text{s.t.} & y_1 & + & 3y_2 & = & 3 \\ & 2y_1 & + & 3y_2 & \geq & -1 \\ & \multicolumn{5}{l}{y_1 \text{ urs., } y_2 \geq 0.} \end{array}$$

Uniqueness and dual of dual

- Is the dual LP unique?

Proposition 1

For any LP, there is a unique dual LP.

- What is the dual of a dual LP?

Proposition 2

For any LP, the dual LP of its dual LP is itself.

Road map

- Primal-dual pairs.
- **Duality theorems**.
- Shadow prices.

Duality theorems

- Duality provides many interesting properties.
- We will illustrate these properties with the **normal max** and **normal min** pair:

$$\begin{array}{rl} \max & cx \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min & yb \\ \text{s.t.} & yA \geq c \\ & y \geq 0 \end{array} \tag{1}$$

  - $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{m \times 1}$, and $c \in \mathbb{R}^{1 \times n}$,
  - $x \in \mathbb{R}^{n \times 1}$ and $y \in \mathbb{R}^{1 \times m}$.
- It can be shown that all the properties we introduce apply to general primal-dual pairs.

Weak duality

- We first show that the dual LP indeed provides an upper bound of the primal LP.

Proposition 3 (Weak duality)

For the LPs defined in (1), if x and y are primal and dual feasible, then $cx \leq yb$.

Proof. Since $x \geq 0$ and $yA \geq c$, we have $yAx \geq cx$. Similarly, $y \geq 0$ and $Ax \leq b$ imply $yAx \leq yb$. Combining $yAx \geq cx$ and $yAx \leq yb$ proves the theorem. □

The dual optimal solution

- If we have solved the primal LP, may we find the dual optimal solution?

Proposition 4 (Dual optimal solution)

For the LPs defined in (1), if $\bar{x}$ is primal optimal with basis B, then $\bar{y} = c_B A_B^{-1}$ is dual optimal.

- This proposition tells us that, once we solve one of the two LPs, the other one can be solved immediately.
- To prove this proposition, we need two lemmas.

The dual optimal solution

Lemma 1

If $\bar{x}$ and $\bar{y}$ are primal and dual feasible and $c\bar{x} = \bar{y}b$, then $\bar{x}$ and $\bar{y}$ are primal and dual optimal.

Proof. For all dual feasible y, we have $c\bar{x} \leq yb$ by weak duality. But we are given that $c\bar{x} = \bar{y}b$, so we have $\bar{y}b \leq yb$ for all dual feasible y. This just tells us that $\bar{y}$ is dual optimal. For $\bar{x}$ it is the same. □

The dual optimal solution

Lemma 2

If B is the primal optimal basis,, then $c_B A_B^{-1}$ is the reduced cost of primal slacks.

Proof. The reduced cost for nonbasic variables is $\bar{c} = c_B A_B^{-1} A_N - c_N$. Let's extend this definition also to basic variables and say that a basic variable has $0 = c_B A_B^{-1} A_B - c_B$ as its reduced cost. With this, we can define

$$\tilde{c} = c_B A_B^{-1} A - c$$

as the reduced cost for all variables. For the ith primal slack x_{n+i}, we know $c_{n+i} = 0$ and $A_{n+i} = e_i$, where e_i is a column vector whose ith element is 1 and all others are 0. Therefore,

$$\tilde{c}_{n+i} = c_B A_B^{-1} A_i - c_{n+i} = c_B A_B^{-1} e_i - 0 = (c_B^T A_B^{-1})_i.$$

As this applies to all i, the statement follows. □

The dual optimal solution

- Now we are ready for the theorem for dual optimal solutions:

  For the LPs defined in (1), if $\bar{x}$ is primal optimal with basis B, then $\bar{y} = c_B A_B^{-1}$ is dual optimal.

- First we show that $\bar{y}$ is dual feasible:
  - As B is the primal optimal basis, $c_B A_B^{-1} A_N - c_N \geq 0$ (otherwise B is not optimal) and thus $c_B A_B^{-1} A_N \geq c_N$. As $c_B A_B^{-1} A_B = c_B$, we have

    $$c_B A_B^{-1} [A_B \ A_N] \geq [c_B \ c_N] \qquad \text{or} \qquad c_B A_B^{-1} A \geq c,$$

    which is exactly $\bar{y} A \geq c$.
  - By Lemma 2, we know $\bar{y}$ is the reduced cost for primal slacks. As B is primal optimal, we know the reduced cost for all variables must be nonnegative, which means $\bar{y} \geq 0$.
- Since $\bar{y}$ is dual feasible and $\bar{y} b = c_B A_B^{-1} b = c_B x_B = c\bar{x}$, we know $\bar{y}$ is dual optimal by Lemma 1.

Strong duality

- The fact that $\bar{y} = c_B A_B^{-1}$ is dual optimal implies strong duality:

Proposition 5 (Strong duality)

$\bar{x}$ and $\bar{y}$ are primal and dual optimal if and only if $\bar{x}$ and $\bar{y}$ are primal and dual feasible and $c^T \bar{x} = \bar{y}^T b$.

Proof. To prove this if-and-only-if statement:

- ($\Leftarrow$): By Lemma 1.
- ($\Rightarrow$): As $\bar{y}$ is dual optimal, $\bar{y}b = c_B A_B^{-1} b = c_B x_B = c\bar{x}$. Note that even if there are multiple optimal solutions to the dual LP, $\bar{y}$ can only result in the same objective value as $c_B A_B^{-1}$ does because $c_B A_B^{-1}$ is also dual optimal. □

Implications of strong duality

- Strong duality is **stronger** than weak duality.
  - Weak duality says that the dual LP provides a bound.
  - Strong duality says the bound is tight, i.e., cannot be improved.
- Given the result of one LP, we may predict the result of its dual:

Primal	Dual		
	Infeasible	Unbounded	Finitely optimal
Infeasible	$\surd$	$\surd$	$\times$
Unbounded	$\surd$	$\times$	$\times$
Finitely optimal	$\times$	$\times$	$\surd$

  - $\surd$ means possible, $\times$ means impossible.
  - Primal unbounded $\Rightarrow$ no upper bound $\Rightarrow$ dual infeasible.
  - Primal finitely optimal $\Rightarrow$ finite objective $\Rightarrow$ dual finitely optimal.
  - If primal is infeasible, the dual may still be infeasible.

Road map

- Primal-dual pairs.
- Duality theorems.
- **Shadow prices**.

A resource allocation problem

- Suppose we produce tables and chairs with wood and labors. In total we have six units of wood and six labor hours.
  - Each table, which can be sold at \$3, requires two units of wood and one labor hour.
  - Each chair, which can be sold at \$1, requires one unit of wood and two labor hours.

  How may we formulate an LP to maximize our sales revenue?
- The decision variables are

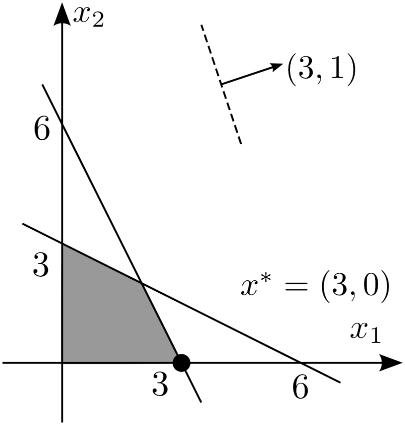
$$x_1 = \text{number of tables produced}$$
$$x_2 = \text{number of chairs produced.}$$

A resource allocation problem

- The LP:

$$\begin{array}{rrcrcr} \max & 3x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 6 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & & & x_i \geq 0 & \forall i = 1, 2. & \end{array}$$

- The optimal solution is $x^* = (3, 0)$.



"What-if" questions

- In practice, people often ask "**what-if**" questions:
  - What if the unit price of chairs becomes $2?
  - What if each table requires three unit of wood?
  - What if we have ten units of woord?
- Why what-if questions?
  - Parameters may fluctuate.
  - Estimation of parameters may be inaccurate.
  - Looking for ways to improve the business.
- For realistic problems, what-if questions can be hard.
  - Even though it may be just a tiny modification of one parameter, it is hard to imagine how the optimal solution will be affected.
- The tool for answering what-if questions is <u>sensitivity analysis</u>.

"What-if" questions

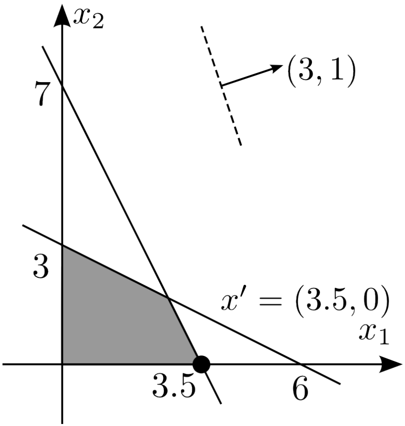
- In general, what-if questions can always be answered by formulating and solving a new optimization problem.
- But this may be too time consuming!
  - We will see that **duality** helps.
- Here we want to introduce only one type of what-if question: What if I have **additional units** of a certain resource?
- Consider the following scenario:
  - One day, a salesperson enters your office and wants to offer you one additional unit of wood at \$1. Should you accept or reject?

One more unit of wood

- To answer this question, you may formulate a new LP:

$$
\begin{array}{rrcrcl}
\max & 3x_1 & + & x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \leq & 7 \\
 & x_1 & + & 2x_2 & \leq & 6 \\
 & & & x_i \geq 0 & \forall i = 1, 2. &
\end{array}
$$

- As the new objective value $z' = 3 \times 3.5 = 10.5$ is larger than the old objective value $z^* = 9$ by 1.5, it is good to accept the offer.
- We earn \$0.5 as our net benefit.

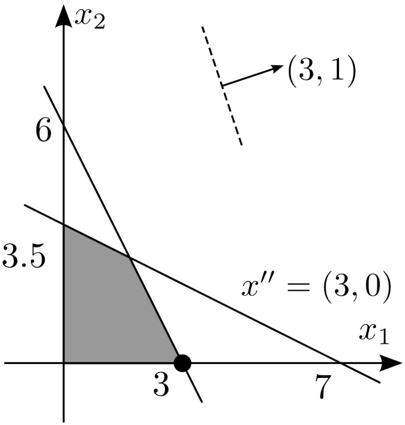


One more labor hour

- Suppose instead of offering one addition unit of wood, the salesperson offers one additional labor hour at 1.

$$
\begin{array}{rrcrcr}
\max & 3x_1 & + & x_2 & & \\
\text{s.t.} & 2x_1 & + & x_2 & \leq & 6 \\
 & x_1 & + & 2x_2 & \leq & 7 \\
 & & & x_i \geq 0 & & \forall i = 1, 2.
\end{array}
$$

- It is not worthwhile to buy it: The objective value does not increase.



Shadow prices

- So for this environment, we know for each resource there is a **maximum amount of price** we are willing to pay for one additional unit.
  - For wood, this price is \$1.5.
  - For labor hours, this price is \$0.
- This motivates us to define **shadow prices** for each constraint:

Definition 1

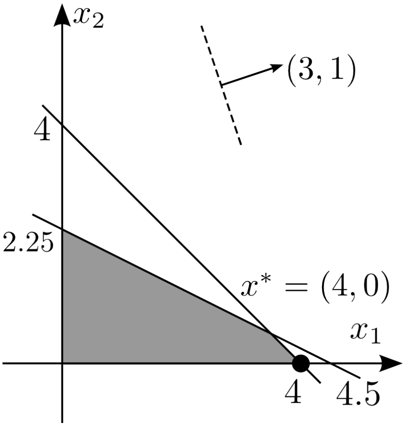
For an LP that has an optimal solution, the shadow price of a constraint is the amount of objective value improved when the RHS of that constraint is increased by 1, assuming the current optimal basis remains optimal.

  - For max LPs, improvement means increasing the objective value.
  - For min LPs, improvement means decreasing the objective value.

Shadow prices

- So for our table-chair example, the shadow prices for constraints 1 and 2 are 1.5 and 0, respectively.
- Note that we **assume** that the current optimal basis does not change.
- Consider another example:

$$
\begin{array}{rrcrcr}
z^* = \max & 3x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
 & x_1 & + & 2x_2 & \leq & 4.5 \\
 & x_i \geq 0 & \forall i = 1, 2. & & &
\end{array}
$$

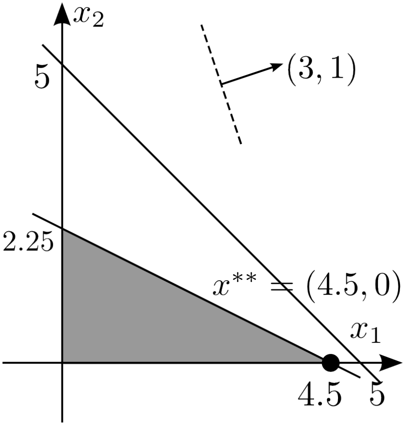


Shadow prices

- If we want to find the shadow price of constraint 1, we may try to solve a new LP:

$$\begin{array}{rrcrcr} z^{**} = \max & 3x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 5 \\ & x_1 & + & 2x_2 & \leq & 4.5 \\ & x_i \geq 0 & & \forall i = 1, 2. & & \end{array}$$

- Though $z^{**} = 13.5$ and $z^{*} = 12$, the shadow price is **not** 1.5!
- By definition, it is $15 - 12 = 3$. Why?
- So shadow prices measure the **rate** of improvement.



Properties of shadow prices

- As a shadow price measures how the objective value is **improved**, its sign is determined based on how the feasible region changes:

Proposition 6

Shadow prices are

- *nonnegative for less-than-or-equal-to constraints,*
- *nonpositive for greater-than-or-equal-to constraints, and*
- *urs. for equality constraints.*

- Less-than-or-equal-to constraint $\Rightarrow$ increasing RHS (weakly) enlarges the feasible region $\Rightarrow$ we can do (weakly) better $\Rightarrow$ the objective value (weakly) increases $\Rightarrow$ nonnegative shadow price.

Properties of shadow prices

- If shifting a constraint does not affect the optimal solution, the shadow price must be zero:

Proposition 7

Shadow prices are 0 for constraints that are not binding at the optimal solution.

  - Not all binding constraints has nonzero shadow prices. Why?
- Now we know finding shadow prices allows us to answer the questions regarding additional units of resources. But how to find all shadow prices?
  - Let m be the number of constraints.
  - Is there a better way than solving m LPs?

Dual optimal solution provide shadow prices

- Duality helps!

Proposition 8

For a maximization LP, shadow prices equal the values of dual variables in the dual optimal solution.

Proof. Let B be the old optimal basis and $z = c_B A_B^{-1} b$ be the old objective value. If b_1 becomes $b_1' = b_1 + 1$, then z becomes

$$z' = c_B A_B^{-1} \left(b + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = z + \left(c_B A_B^{-1} \right)_1 .$$

So the shadow price of constraint 1 is $(c_B A_B^{-1})_1$. In general, the shadow price of constraint i is $(c_B A_B^{-1})_i$. as $c_B A_B^{-1}$ is the dual solution, the proof is complete. □

Shadow prices for minimization LPs

- Therefore, to find m shadow prices, we do not need to solve m new LPs. It suffices to solve **only one** LP, the dual LP.
- For minimization LPs, simply negate the dual optimal solution:

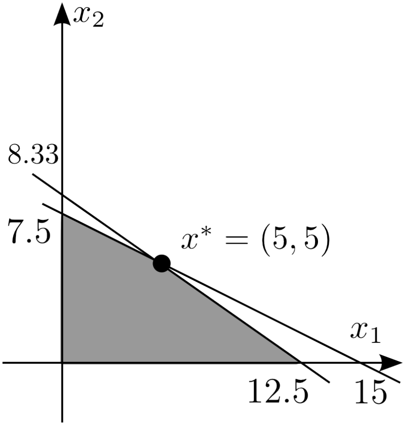
Proposition 9

For a minimization LP, shadow prices equal the negation of the values of dual variables in the dual optimal solution.

An example

- What are the shadow prices of the two functional constraints?

$$
\begin{array}{rrcrcl}
\max & 3x_1 & + & 5x_2 & & \\
\text{s.t.} & 2x_1 & + & 3x_2 & \leq & 25 \\
 & x_1 & + & 2x_2 & \leq & 15 \\
 & & & x_i \geq 0 & & \forall i = 1, 2.
\end{array}
$$



An example

- We solve the dual LP

$$\begin{array}{rrcrcl} \max & 25y_1 & + & 15y_2 & & \\ \text{s.t.} & 2y_1 & + & y_2 & \geq & 3 \\ & 3y_1 & + & 2y_2 & \geq & 5 \\ & & y_i \geq 0 & \forall i = 1, 2. & & \end{array}$$

  The dual optimal solution is $y^* = (1, 1)$.

- So shadow prices are 1 and 1 for primal constraints 1 and 2, respectively.