Operations Research, Spring 2014
Final Exam

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

Name: ____________________ Student ID: ____________________

Note 1. You do not need to return these problem sheets. Write down all your answers on the answer sheets provided to you.

Note 2. In total there are 110 points. If you get more than 100, your official score will only be 100.

1. (15 points; 5 points each) Consider a linear program

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 6 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

Prove that the unique optimal solution is $(x_1^*, x_2^*) = (2, 2)$ with the following approaches:

(a) The graphical approach.
(b) The simplex method.
(c) The KKT condition.

Note. Please answer Parts (a), (b), and (c) separately.

2. (10 points; 5 points each) Consider the network in Figure 1, where arc weights are capacities.

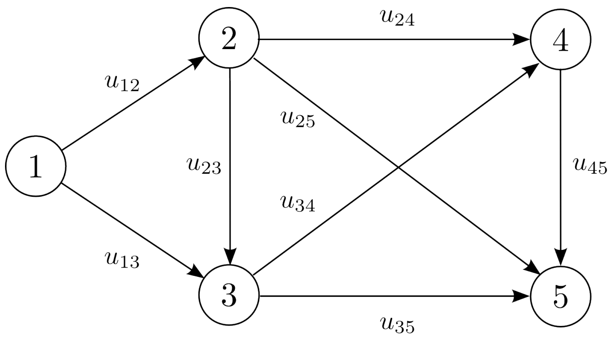

Figure 1: Network for maximum flow problem

(a) Formulate a linear program that finds a maximum flow from node 1 to node 5. DO NOT try to find a maximum flow.
(b) Ignore Part (a). Suppose that at most w_3, w_4, and w_5 units of flow are allowed to pass through nodes 3, 4, and 5, respectively. For such a nonstandard maximum flow problem with node capacities, convert it into an equivalent standard maximum flow problem with only arc capacities. Drawing a figure to demonstrate your solution is preferred. Again, DO NOT try to find a maximum flow.

3. (10 points) Draw a branch-and-bound tree to solve

$$\begin{aligned} \min \quad & x_1 + 2x_2 + x_3 + 3x_4 \\ \text{s.t.} \quad & 4x_1 + x_2 + 2x_3 + 2x_4 \geq 7. \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 4. \end{aligned}$$

4. (10 points) Use the simplex method with the smallest index rule to solve

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 - x_4 = 12 \\ & 2x_1 + x_2 + x_3 \leq 8 \\ & x_i \geq 0 \quad \forall i = 1, ..., 4. \end{aligned}$$

Write down all the iterations to get full credits.

5. (15 points; 5 points each) In a concert hall, there are n seats in total. As a manager, you may divide the n seats into at most four classes. The price for a seat in class i has been determined to be p_i, $i = 1, ..., 4$. Your decision variables are x_i, the number of seats in class i, $i = 1, ..., 4$. Considering the structure of the concert hall, you know the maximum numbers of seats in classes 1 and 2 are m_1 and m_2, respectively. Moreover, the number of seats in class i cannot be greater than that in class j if $i < j$.

   (a) Suppose the demand of seats in class i is fixed to d_i and will not be affected by anything. Therefore, the sales quantity of class i is $\min\{d_i, x_i\}$. Formulate a nonlinear program that maximizes your total sales revenue WITHOUT using a minimum function.

   (b) Continue from Part (a) and assume that $d_i = \infty$ for all i, i.e., there are unlimited demands for all classes, and the sales quantity of class i is exactly x_i. Determine whether the nonlinear program is a convex program. Explain why.

   (c) Ignore Parts (a) and (b). Suppose that the demand for class i is a random variable D_i. We assume that D_i is continuous and nonnegative and D_is are independent. The expected sales quantity of class i is thus $\mathbb{E}\min\{D_i, x_i\}$. For a nonlinear program that maximizes your total expected sales revenue, determine whether it is a convex program. Explain why.

6. (20 points; 5 points each) In a factory, one decision maker faces n jobs to be assigned to m machines. The *processing time* of job j is q_j, i.e., it takes q_j units of time to complete job j. The processing time is independent of the jobs to be processed before or after it. The decision maker must assign each job to a machine without splitting a job. After the assignment has been determined, for each machine the *completion time* is the sum of the processing times of those jobs assigned to it. Then the *makespan* is the maximum completion time.

   For example, suppose there are three machines and ten jobs. For jobs 1, 2, ..., and 10, their processing times q_js are 2, 2, 2, 5, 5, 6, 6, 6, 8, and 9, respectively. One not-so-good schedule is to assign jobs 1 to 4 to machine 1, jobs 5 to 7 to machine 2, and jobs 8 to 10 to machine 3. The resulting completion times of the three machines are 11, 17, and 23, respectively, and thus the makespan is 23. One better schedule is to assign jobs 1, 4, 5, and 6 to machine 1, jobs 2, 7, and 9 to machine 2, and jobs 3, 8, and 10 to machine 3. The completion times of the three machines are 18, 16, and 17, respectively. The makespan 18 is better than 23. However, we have no idea whether this schedule is an optimal schedule.

   (a) Formulate a linear integer program that can minimizes the makespan of a general problem with n jobs, m machines, and processing times q_1, q_2, ..., and q_n. Please note that you should optimize the general problem rather than the example above.

   (b) Consider the linear relaxation of the given example with three machines and ten jobs. For the relaxed problem, what is the minimum makespan? Explain why.

   (c) Ignore Part (b) and continue from Part (a). Suppose now no machine can be assigned more than K jobs, formulate a linear integer program that can minimizes the makespan.

   (d) Ignore Parts (b) and (c) and continue from Part (a). Suppose now at most two machines can be assigned more than K jobs, formulate a linear integer program that can minimizes the makespan.

7. (15 points; 5 points each) In a market, manufacturer 1 sells product 1 to a retailer at wholesale price w_1, who then sells to end consumers at retail price p_1. At the same market, manufacturer 2 sells a similar product, product 2, to end consumers directly at retail price p_2. For each manufacturer, the production cost of the product is assumed to be 0. The sales quantity of product i is

$$1 - p_i + \theta p_{3-i}, \quad i = 1, 2,$$

where $\theta \in [0, 1)$ is the degree of substitutability. Each of manufacturer 1, manufacturer 2, and the retailer sets w_1, p_2, and p_1 to maximize its own profit. In this three-player game, in period 1 manufacturer 1 sets w_1; then in period 2 manufacturer 2 and the retailer set p_2 and p_1 simultaneously.

   (a) Given the wholesale price w_1 announced in period 1, find the equilibrium retail prices p_1 and p_2 in period 2 as functions of w_1.

   (b) Find the equilibrium wholesale price w_1 in period 1.

   (c) Prove or disprove that $p_1 > p_2$ for all $\theta \in [0, 1)$ in equilibrium.

8. (15 points) A volleyball team with fifteen players is deciding whether to participate in a contest. They have agreed that, to participate in the contest, there must be at least eight team members who are willing to go. Moreover, among the six players who play better, at least three of them must go. If any of the two conditions is not satisfied, they will choose not to register for the contest. To understand each team member's true preferences, the captain decides to ask each of them to write down "Yes, I would like to go" or "No, I prefer not to go" on a paper contained in a sealed envelope. Therefore, each team member will make her own decision without knowing any other member's decision. After team members all submit their envelopes, the caption will make the final decision according to the two requirements listed above. If they decide to join the contest, exactly those who say yes will be officially registered. In other words, if one says no, then in any case she will not join the contest; if one says yes, she will join the contest when the two conditions are satisfied.

   (a) (5 points) Is this a static game or a dynamic game? Briefly explain.

   (b) (10 points) For each of those six players who play better, joining the contest gives her 10 as the payoff. For each of those nine players who play not so well, joining the contest gives her 6 as the payoff. For all of them, not joining the contest gives 0 as the payoff. Find all the Nash equilibria.[1]

[1] We are only interested in pure-strategy Nash equilibria.