IM 2010: Operations Research, Spring 2014

Introduction to Linear Programming

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A promise is a promise

- If you produce foods, what are important in getting an order from restaurants and retailers?
  - Customers ask "When may I get them?" and "How much may I get?"
  - You need to give accurate answers immediately.
  - You need to **promise** and keep your promise.
- Why difficult?
  - You have more than 8000 customers sharing your capacity and inventory.
  - Once you promise one customer, you need to immediately **update** the availability information that are needed elsewhere.
  - And updating requires a lot of **planning** and calculations.
- Read the application vignette in Section 3.1 and the article on CEIBA.

Introduction

- We need a very powerful way of planning.
- In the next five weeks, we will study **Linear Programming** (LP).
  - It is used a lot in practice.
  - It also provides important theoretical properties.
  - It is good starting point for all OR subjects.
- We will study:
  - What kind of practical problems can be solved by LP.
  - How to formulate a problem as an LP.
  - How to solve an LP.
  - Any many more.
- Read Chapter 3 for this lecture!
  - Read Sections 3.1 to 3.3 thoroughly.
  - Read Section 3.4 for many examples that we do not have time to cover.
  - Read Section 3.5 after the TA session on March 3.
  - Skip Section 3.6.

Road map

- **Terminology**.
- The graphical approach.
- Three types of LPs.
- Simple LP formulations.
- Compact LP formulations.

Introduction

- Linear Programming is the process of formulating and solving **linear programs** (also abbreviated as LP).
- An LP is a mathematical program with some special properties.
- Let's first introduce some concepts of mathematical programs.

Basic elements of a program

- In general, any mathematical program can be expressed as

$$
\begin{array}{lllr}
\min & f(x) & & (\textbf{objective function}) \\
\text{s.t.} & g_i(x) \leq b_i & \forall i = 1, ..., m & (\textbf{constraints}) \\
 & x_j \in \mathbb{R} & \forall j = 1, ..., n. & (\textbf{decision variable})
\end{array}
$$

  - There are m constraints and n variables.
  - $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ is a vector.
  - $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ are all real-valued functions.
  - Mostly we will omit $x_j \in \mathbb{R}$.

Transformation

- How about a maximization objective function?
  - $\max f(x) \Leftrightarrow \min -f(x)$.
- How about "=" or "$\geq$" constraints?
  - $g_i(x) \geq b_i \Leftrightarrow -g_i(x) \leq -b_i$.
  - $g_i(x) = b_i \Leftrightarrow g_i(x) \leq b_i$ and $g_i(x) \geq b_i$, i.e., $-g_i(x) \leq -b_i$.

$$\begin{array}{rrcrcr} \max & x_1 & - & x_2 & & \\ \text{s.t.} & -2x_1 & + & x_2 & \geq & -3 \\ & x_1 & + & 4x_2 & = & 5. \end{array} \quad \Leftrightarrow \quad \begin{array}{rrcrcr} \min & -x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & - & x_2 & \leq & 3 \\ & x_1 & + & 4x_2 & \leq & 5 \\ & -x_1 & - & 4x_2 & \leq & -5. \end{array}$$

Sign constraints

- For some reasons that will be clear in the next week, we distinguish between two kinds of constraints:
  - **Sign constraints**: $x_i \geq 0$ or $x_i \leq 0$.
  - **Functional constraints**: all others.
- For a variable x_i:
  - It is **nonnegative** if $x_i \geq 0$.
  - It is **nonpositive** if $x_i \leq 0$.
  - It is **unrestricted in sign** (urs.) or **free** if it has no sign constraint.

Feasible solutions

- For a mathematical program:
  - A **feasible solution** satisfies all the constraints.
  - An **infeasible solution** violates at least one constraint.

$$
\begin{array}{rrcrcr}
\min & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 12 \\
 & x_1 & - & 2x_2 & \geq & -8 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0.
\end{array}
$$

- Feasible?
  - $x^1 = (2, 3)$.
  - $x^2 = (6, 0)$.
  - $x^3 = (6, 6)$.

Feasible region and optimal solutions

- The **feasible region** (or **feasible set**) is the set of feasible solutions.
  - The feasible region may be empty.
- An **optimal solution** is a feasible solution that:
  - Attains the largest objective value for a maximization problem.
  - Attains the smallest objective value for a minimization problem.
  - In short, no feasible solution is better than it.
- An optimal solution may not be unique.
  - There may be **multiple** optimal solutions.
  - There may be **no** optimal solution.

Binding constraints

- At a solution, a constraint may be **binding**:[1]

Definition 1

Let $g(\cdot) \leq b$ be an inequality constraint and $\bar{x}$ be a solution. $g(\cdot)$ is binding at $\bar{x}$ if $g(\bar{x}) = b$.

  - An inequality is **nonbinding** at a point if it is strict at that point.
  - An equality constraint is always binding at any feasible solution.
- Some examples:
  - $x_1 + x_2 \leq 10$ is binding at $(x_1, x_2) = (2, 8)$.
  - $2x_1 + x_2 \geq 6$ is nonbinding at $(x_1, x_2) = (2, 8)$.
  - $x_1 + 3x_2 = 9$ is binding at $(x_1, x_2) = (6, 1)$.

[1] Binding/nonbinding constraints are also called **active**/inactive constraints.

Strict constraints?

- An inequality may be **strict** or **weak**:
  - It is strict if the two sides cannot be equal. E.g., $x_1 + x_2 > 5$.
  - It is weak if the two sides may be equal. E.g., $x_1 + x_2 \geq 5$.
- A "practical" mathematical program's inequalities are **all weak**.
  - With strict inequalities, an optimal solution may not be attainable!
  - What is the optimal solution of

$$\begin{array}{rl} \min & x \\ \text{s.t.} & x > 0? \end{array}$$

- Think about budget constraints.
  - You want to spend \$500 to buy several things.
  - Typically, you cannot spend more than \$500.
  - But you can spend exactly \$500.

Linear Programs

- For a mathematical program

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq b_i \quad \forall i = 1, ..., m, \end{aligned}$$

  if f and g_is are all **linear** functions, it is an LP.

- In general, an LP can be expressed as

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} A_{ij} x_j \leq b_i \quad \forall i = 1, ..., m. \end{aligned}$$

  - A_{ij}s: the **constraint coefficients**.
  - b_is: the right-hand-side values (**RHS**).
  - c_js: the **objective coefficients**.

- Or expressed by matrices:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b. \end{aligned}$$

  - $A \in \mathbb{R}^{m \times n}$.
  - $b \in \mathbb{R}^m$.
  - $c \in \mathbb{R}^n$.
  - $x \in \mathbb{R}^n$.

Summary

- The decision variables, objective function, and constraints.
- Functional and sign constraints.
- Feasible solutions and optimal solutions.
- Binding constraints.

Road map

- Terminology.
- **The graphical approach**.
- Three types of LPs.
- Simple LP formulations.
- Compact LP formulations.

Graphical approach

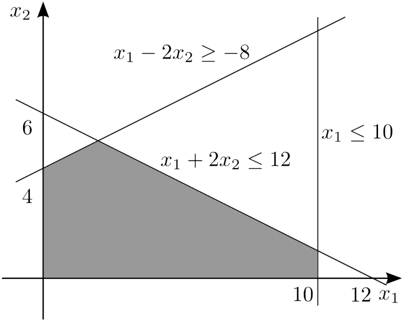
- For LPs with only two decision variables, we may solve them with the **graphical approach**.
- Consider the following example:

$$
\begin{array}{rrcrcl}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 12 \\
& x_1 & - & 2x_2 & \geq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0.
\end{array}
$$

Graphical approach

- Step 1: Draw the feasible region.
  - Draw each constraint one by one, and then find the intersection.

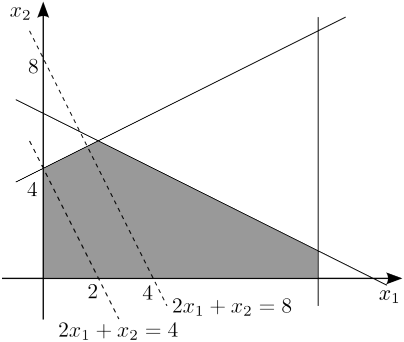
$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 12 \\
& x_1 & - & 2x_2 & \geq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0.
\end{array}
$$



Graphical approach

- Step 2: Draw some **isoquant lines**.
  - A line such that all points on it result in **the same** objective value.
  - Also called **isoprofit** or **isocost** lines when it is appropriate.
  - Also called **indifference lines** (curves) in Economics.

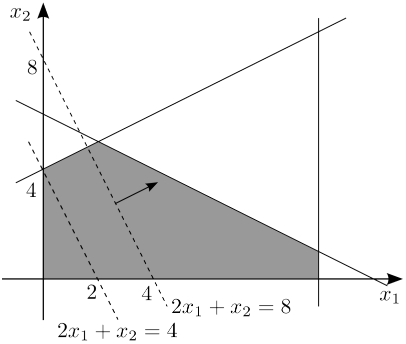
$$
\begin{array}{lrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 12 \\
 & x_1 & - & 2x_2 & \geq & -8 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0.
\end{array}
$$



Graphical approach

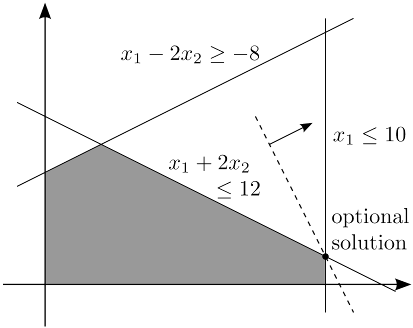
- Step 3: Indicate the direction to push the isoquant line.
  - The direction that **decreases**/increases the objective value for a **minimization**/maximization problem.

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 12 \\
 & x_1 & - & 2x_2 & \geq & -8 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0.
\end{array}
$$



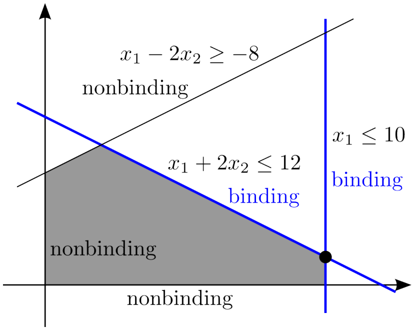
Graphical approach

- Step 4: Push the isoquant line to the "end" of the feasible region.
  - Stop when any further step makes all points on the isocost line infeasible.



Graphical approach

- Step 5: Identify the binding constraints at the optimal solution.



Graphical approach

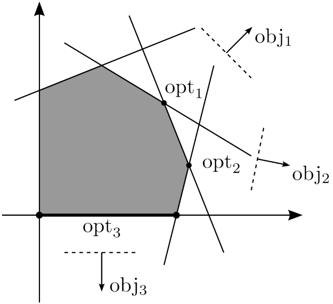
- Step 6: Set the binding constraints to equalities and then solve the linear system for an optimal solution.
  - In the example, the binding constraints are $x_1 \leq 10$ and $x_1 + 2x_2 \leq 12$. Therefore, we solve

$$\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 1 & 2 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 1 \end{array}\right]$$

    and obtain an optimal solution $(x_1^*, x_2^*) = (10, 1)$.
- Step 7: Plug in the optimal solution obtained into the objective function to get the associated objective value.
  - In the example, $2x_1^* + x_2^* = 21$.

Where to stop pushing?

- Where we push the isoquant line, where will be stop at?
- Intuitively, we **always** stop at a "**corner**" (or an edge).



- Is this intuition still true for LPs with more than two variables?
- Yes! With a more rigorous definition of "corners".

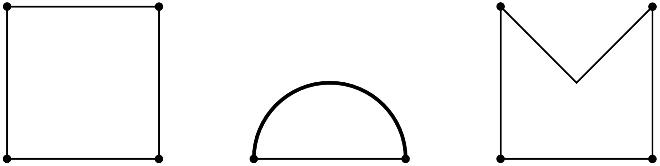
Extreme points

- We need to first define **extreme points** for a set:[2]

Definition 2 (Extreme points)

For a set $S \subseteq \mathbb{R}^n$, a point x is an extreme point if there does not exist a three-tuple (x^1, x^2, λ) such that $x^1 \in S \setminus \{x\}$, $x^2 \in S \setminus \{x\}$, $\lambda \in (0, 1)$, and

$$x = \lambda x^1 + (1 - \lambda)x^2.$$



[2] In the textbook, extreme points are called corner-point solutions.

Optimality of extreme points

- For any LP, we have the following fact.

Proposition 1

For any LP, if there is an optimal solution, there is an extreme point optimal solution.

  - It is not saying that "if a solution is optimal, it is an extreme point!"
- This property will be very useful when we develop a method for solving general LPs!

Graphical approach: Summary

- Six steps:
  - Step 1: Feasible region.
  - Step 2: Isoquant line.
  - Step 3: Direction to push (i.e., the improving direction).
  - Step 4: Push!
  - Step 5: Binding constraints at an optimal solution.
  - Step 6: An optimal solution and the associated objective value.
- Make your graph clear and in the right scale to avoid mistakes.

Road map

- Terminology.
- The graphical approach.
- **Three types of LPs**.
- Simple LP formulations.
- Compact LP formulations.

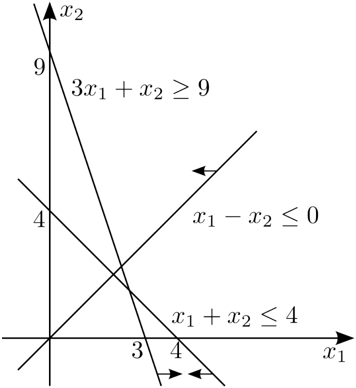
Three types of LPs

- For any LPs, it must be one of the following:
  - Infeasible.
  - Unbounded.
  - Finitely optimal (having an optimal solution).
- A finitely optimal LP may have:
  - A unique optimal solution.
  - Multiple optimal solutions.

Infeasibility

- An LP is **infeasible** if its feasible region is empty.

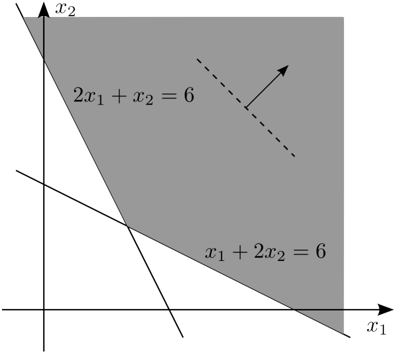
$$
\begin{array}{rrcrcl}
\min & 3x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
& 3x_1 & + & x_2 & \geq & 9 \\
& x_1 & - & x_2 & \leq & 0.
\end{array}
$$



Unboundedness

- An LP is **unbounded** if for any feasible solution, there is another feasible solution that is better.

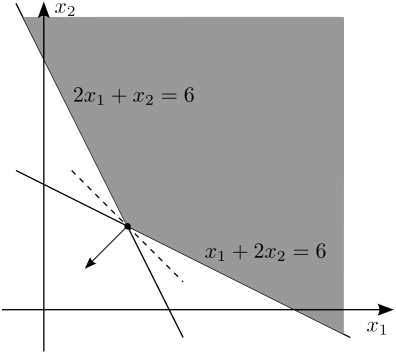
$$
\begin{array}{rrcrcl}
\max & x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\
 & 2x_1 & + & x_2 & \geq & 6.
\end{array}
$$



Unboundedness

- Note that an unbounded feasible region **does not imply** an unbounded LP!
  - Is it necessary?

$$
\begin{array}{rrcrcl}
\min & x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\
 & 2x_1 & + & x_2 & \geq & 6.
\end{array}
$$

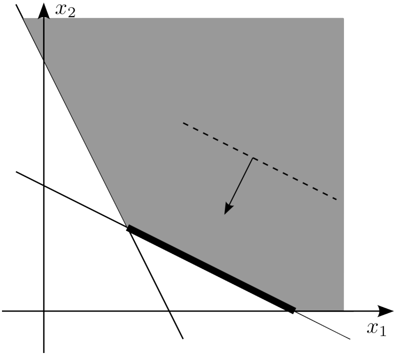


- If an LP is neither infeasible nor unbounded, it is **finitely optimal**.

Multiple optimal solutions

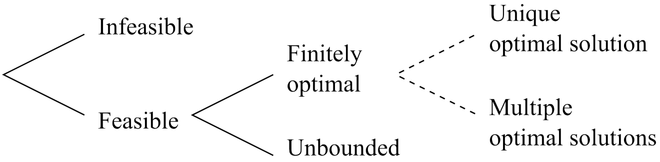
- A linear program may have **multiple** optimal solutions.

$$\begin{array}{rrcrcl} \min & x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\ & 2x_1 & + & x_2 & \geq & 6 \\ & & & x_2 & \geq & 0. \end{array}$$



- If the slope of the isoquant line is identical to that of one constraint, will we always have multiple optimal solutions?

Summary



- In solving an LP (or any mathematical program) in practice, we only want to find **an** optimal solution, not all.
  - All we want is to make an optimal decision.

Road map

- Terminology.
- The graphical approach.
- Three types of LPs.
- **Simple LP formulations**.
- Compact LP formulations.

Introduction

- It is important to learn how to model a practical situation as an LP.
  - Once you do so, you have "**solved**" the problem.
- This process is typically called **LP formulation** or **modeling**.
- Here we will give you two examples of LP formulation.
  - We will do more in lectures, TA sessions, homework, case assignments, exams, and (most likely) the final project.
  - Practice makes perfect!
- Then we formulate large-scale problems with **compact formulations**.

A product mix problem

- We produce several products to sell.
- Each product requires some resources. **Resources are limited**.
- We want to maximize the total sales revenue with available resources.

Problem description

- We produce desks and tables.
    - Producing a desk requires three units of wood, one hour of labor, and 50 minutes of machine time.
    - Producing a table requires five units of wood, two hours of labor, and 20 minutes of machine time.
- We may sell everything we produce.
- For each day, we have
    - Two hundred workers that each works for eight hours.
    - Fifty machines that each runs for sixteen hours.
    - A supply of 3600 units of wood.
- Desks and tables are sold at \$700 and \$900 per unit, respectively.

DFSI: (1) Define variables

- What do we need to decide?
- Let

$$\begin{aligned} x_1 &= \text{number of desks produced in a day and} \\ x_2 &= \text{number of tables produced in a day.} \end{aligned}$$

- With these variables, we now try to **express** how much we will earn and how many resources we will consume.

DFSI: (2a) Formulate the objective function

- We want to maximize the total sales revenue.
- Given our variables x_1 and x_2, the sales revenue is $700x_1 + 900x_2$.
- The objective function is thus

$$\max \; 700x_1 + 900x_2.$$

DFSI: (2b) Formulate constraints

- For each **restriction** or **limitation**, we write a constraint.
- Summarizing data into a table typically helps:

Resource	Consumption per		Total supply
	Desk	Table	
Wood	3 units	5 units	3600 units
Labor hour	1 hour	2 hours	200 workers $\times$ 8 hr/worker $= 1600$ hours
Machine time	50 minutes	20 minutes	50 machines $\times$ 16 hr/machine $= 800$ hours

- The supply of wood is limited: $3x_1 + 5x_2 \leq 3600$.
- The number of labor hours is limited: $x_1 + 2x_2 \leq 1600$.
- The amount of machine time is limited: $50x_1 + 20x_2 \leq 48000$.
  - Use the same **unit of measurement**!

DFSI: (2c) Complete formulation

- Collectively, our formulation is

$$
\begin{array}{rrcrcrl}
\max & 700x_1 & + & 900x_2 & & & \\
\text{s.t.} & 3x_1 & + & 5x_2 & \leq & 3600 & \text{(wood)} \\
 & x_1 & + & 2x_2 & \leq & 1600 & \text{(labor)} \\
 & 50x_1 & + & 20x_2 & \leq & 48000. & \text{(machine)}
\end{array}
$$

  is that all?[3]
- In any case:
  - **Clearly** define decision variables **in front of** your formulation.
  - Write **comments** after the objective function and constraints.

[3] Think about this and we will discuss it in the lecture.

DFSI: (3 and 4) Solve and interpret

- The optimal solution of this LP is $(884.21, 189.47)$.
- So the interpretation is... to produce 884.21 desks and 189.47 tables?
- Should we impose **integer constraints**?
  - An LP with integer constraints is called an **Integer Program** (IP).
  - Unfortunately, an IP may take an unreasonable time to solve.[4]
- But "producing 884.21 desks and 189.47 tables" is impossible!
  - It still **supports** our decision making.
  - We may **suggest** to produce, e.g., 884 desks and 189 tables.[5]
  - It may not really be optimal.
  - But we spend a very short time to make a good suggestion!

[4] We will discuss IP in details later in this semester.

[5] Why not 885 desks and 190 tables or the other two ways of rounding?

Produce and store!

- When we are making decisions, we may also consider what will happen in the **future**.
- This creates **multi-period** problems.
- In many cases, products produced today may be **stored** and then sold in the future.
  - Maybe daily capacity is not enough.
  - Maybe production is cheaper today.
  - Maybe the price is higher in the future.
- So the production decision must be jointly considered with the **inventory** decision.

Problem description

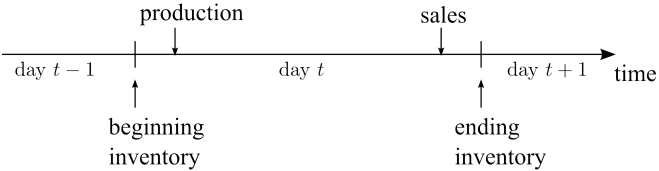
- We produce and sell a product.
- For the coming four days, the marketing manager has promised to fulfill the following amount of demands:
  - Days 1, 2, 3, and 4: 100, 150, 200, and 170 units, respectively.
- The unit production costs are different for different days:
  - Days 1, 2, 3, and 4: \$9, \$12, \$10, and \$12 per unit, respectively.
- The prices are all **fixed**. So maximizing profits is the same as minimizing costs.
- We may store a product and sell it later.
  - The **inventory cost** is \$1 per unit per day.[6]
  - E.g., producing 620 units on day 1 to fulfill all demands costs

$$9 \times 620 + 1 \times 150 + 2 \times 200 + 3 \times 170 = 6640 \text{ dollars.}$$

[6] Where does this inventory cost come from?

Problem description: timing

- Timing:



  - Beginning inventory + production − sales = ending inventory.
  - Inventory costs are calculated according to **ending inventory**.

Variables and objective function

- Let

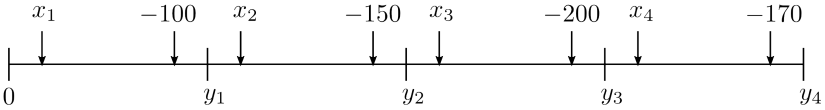
$$x_t = \text{production quantity of day } t, t = 1, ..., 4.$$
$$y_t = \text{ending inventory of day } t, t = 1, ..., 4.$$

  - It is important to specify "ending"!
- The objective function is

$$\min \; 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4.$$

Constraints

- We need to keep an eye on our inventory:



  - Day 1: $x_1 - 100 = y_1$.
  - Day 2: $y_1 + x_2 - 150 = y_2$.
  - Day 3: $y_2 + x_3 - 200 = y_3$.
  - Day 4: $y_3 + x_4 - 170 = y_4$.
- These are typically called **inventory balancing** constraints.
- We also need to fulfill all demands at the moment of sales:
  - $x_1 \geq 100$, $y_1 + x_2 \geq 150$, $y_2 + x_3 \geq 200$, and $y_3 + x_4 \geq 170$.
- Also, production and inventory quantities cannot be negative.

The complete formulation

- The complete formulation is

$$\begin{aligned} \min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 \\ & + y_1 + y_2 + y_3 + y_4 \\ \text{s.t.} \quad & x_1 - 100 = y_1 \\ & y_1 + x_2 - 150 = y_2 \\ & y_3 + x_3 - 200 = y_3 \\ & y_3 + x_4 - 170 = y_4 \\ & x_1 \geq 100 \\ & y_1 + x_2 \geq 150 \\ & y_2 + x_3 \geq 200 \\ & y_3 + x_4 \geq 170 \\ & x_t, y_t \geq 0 \quad \forall t = 1, ..., 4. \end{aligned}$$

- May we simplify the formulation?
- Inventory balancing and nonnegativity together implies demand fulfillment!
  - Day 1: $x_1 - 100 = y_1$ and $y_1 \geq 0$ means $x_1 \geq 100$.
- So the formulation can just be

$$\begin{aligned} \min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 \\ & + y_1 + y_2 + y_3 + y_4 \\ \text{s.t.} \quad & x_1 - 100 = y_1 \\ & y_1 + x_2 - 150 = y_2 \\ & y_3 + x_3 - 200 = y_3 \\ & y_3 + x_4 - 170 = y_4 \\ & x_t, y_t \geq 0 \quad \forall t = 1, ..., 4. \end{aligned}$$

Personnel scheduling

- Numbers of personnel required at an airport vary a lot among different time periods.
- How many people will you hire?
  - Each person works for eight hours **continuously**.
  - They may start their shifts at different time.
  - Demands of personnel ("0–2", "2–4", and "4-6" all need 6 persons):

0–6	6–8	8–10	10–12	12–14	14–16	16–18	18–20	22–24
6	10	15	20	16	24	28	20	10

- LP is used to save more than $6 million annually.
- Read the application vignette in Section 3.4 and the article on CEIBA.

Road map

- Terminology.
- The graphical approach.
- Three types of LPs.
- Simple LP formulations.
- **Compact LP formulations**.

Compact formulations

- Most problems in practice are of **large scales**.
  - The number of variables and constraints are huge.
- Many variables can be grouped together:
  - E.g., x_t = production quantity of day $t, t = 1, ..., 4$.
- Many constraints can be grouped together:
  - E.g., $x_t \geq 0$ for all $t = 1, ..., 4$.
- In modeling large-scale problems, we use **compact formulations** to enhance readability and efficiency.
- We use the following three instruments:
  - Indices $(i, j, k, ...)$.
  - Summation $(\sum)$.
  - For all $(\forall)$.

Compacting the objective function

- The problem:
  - We have four periods.
  - In each period, we first produce and then sell.
  - Unsold products become ending inventories.
  - Want to minimize the total cost.
- Indices:
  - Because things will **repeat in each period**, it is natural to use an index for periods. Let $t \in \{1, ..., 4\}$ be the index of periods.
- The objective function:
  - $\min\ 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4$.
  - $\min\ 9x_1 + 12x_2 + 10x_3 + 12x_4 + \sum_{t=1}^{4} y_t$.
  - If we denote the unit cost on day t as C_t, $t = 1, ..., 4$:

$$\min \sum_{t=1}^{4} (C_t x_t + y_t).$$

Compacting the constraints

- The original constraints:
  - $x_1 - 100 = y_1$, $y_1 + x_2 - 150 = y_2$, $y_2 + x_3 - 200 = y_3$, $y_3 + x_4 - 170 = y_4$.
- Let's denote the demand on day t as D_t, $t = 1, ..., 4$.
- The compact constraint:
  - For $t = 2, ..., 4 : y_{t-1} + x_t - D_t = y_t$.
  - We cannot apply this to day 1 as y_0 is undefined!
- To group the four constraints into one compact constraint, we add an additional decision variable y_0:

$$y_t = \text{ending inventory of day } t, t = 0, ..., 4.$$

- Then the set of inventory balancing constraints are written as

$$y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., 4$$

- Certainly we need to set up the initial inventory: $y_0 = 0$.

The complete compact formulation

- The compact formulation is

$$\begin{aligned} \min \quad & \sum_{t=1}^{4}(C_t x_t + y_t) \\ \text{s.t.} \quad & y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., 4 \\ & y_0 = 0 \\ & x_t, y_t \geq 0 \quad \forall t = 1, ..., 4. \end{aligned}$$

  - **Do not forget** "$\forall t = 1, ..., 4$"! Without that, the formulation is wrong.
  - Nonnegativity constraints for multiple sets of variables can be combined to save some "≥ 0".
- One convention is to:
  - Use **lowercase** letters for variables (e.g., x_t).
  - Use **uppercase** letters for parameters (e.g., C_t).

Parameter declaration

- When creating parameter sets, we write something like

  denote C_t as the unit production cost on day $t, t = 1, ..., 4.$

  - Do not need to specify values, even though we have those values.
  - Need to specify the **range** through **indices**.
- Parameter declarations should be at the beginning of the formulation.
- Parameters and variables are just different.
  - Variables are those to be determined. We do have know there values before we solve the model.
  - Parameters are given with known values.
  - Parameters are **exogenous** and variables are **endogenous**.