Operations Research, Spring 2015
Final Exam

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You do not need to return these problem sheets; write down all your answers on the answer sheets. In total there are 110 points. If you get more than 100, your official score will only be 100.

1. (20 points) Consider the following LP:

$$
\begin{array}{rl}
\min & x_1 - 2x_2 \\
\text{s.t.} & x_1 + x_2 + 3x_3 - x_4 \geq 12 \\
& 2x_1 + 2x_2 + x_3 \leq 8 \\
& x_i \geq 0 \quad \forall i = 1, ..., 4.
\end{array}
$$

   (a) (5 points) Apply the two phase method on this problem. Complete phase I with the smallest index rule to obtain a feasible solution. Write down all the iterations to get full credits.

   (b) (5 points) Continue from Part (a). Complete phase II with the smallest index rule to solve the LP. Write down all the iterations to get full credits.

   (c) (5 points) Suppose that all x_is are subject to integer constraints. If you apply the branch and bound algorithm, write down the two LPs to solve in the second level of the branch-and-bound tree. If there are multiple variables to branch on, branch on the one with the smallest index. Do not solve the two LPs.

   (d) (5 points) Ignore Parts (a) and (b). For the standard form of the LP, write down all the bfs whose $x_3 = 0$.

2. (10 points) Linearize the following program:

$$
\begin{array}{rl}
\min & |x_1 + x_2| - \min\{x_3, x_4 + x_5\} + x_6 \\
\text{s.t.} & 2x_1 + \max\{x_2, x_3\} \leq x_6 \\
& \max\{x_1 + x_2, 4, x_3\} \geq \min\{0, x_4\} \\
& x_i \geq 0 \quad \forall i = 1, ..., 4.
\end{array}
$$

3. (10 points; 5 points each) Answer the following questions.

   (a) For an EPQ problem with monthly demand 2000 units, production rate 1000 per week, holding cost \$1 per unit per month, and setup cost \$500 per lot, what is the EPQ? Assume that each year has 12 months and 48 weeks.

   (b) For a newsvendor problem with a uniform daily demand between 0 and 400, unit purchasing cost \$16, unit sales price \$46, and unit disposal fee \$4, what is the newsvendor quantity?

4. (10 points; 5 points) Consider the following LP

$$
\begin{array}{rl}
\max & 3x_1 + 2x_2 + x_3 \\
\text{s.t.} & x_1 + x_2 + x_3 \leq 10 \\
& 2x_1 + x_2 + 2x_3 \leq 12 \\
& x_1 + 2x_2 \geq 6 \\
& x_i \geq 0 \quad \forall i = 1, ..., 3.
\end{array}
$$

   (a) Find the dual LP.

   (b) Use complementary slackness to prove or disprove that $(x_1, x_2, x_3) = (6, 0, 0)$ is primal optimal.

5. (25 points; 5 points each) As a farmer, you have a field whose area is A m^2. You may split the field into multiple pieces for planting n kinds of plants, plants 1, 2, ..., and n. The unit cost of planting plant i is C_i. Planting plant i requires H_i human hours and R_i liters of pesticide per m^2 per month, respectively. You hire a few farmers and in total you have K human hours per month. You have signed a contract with a long-term pesticide supplier for a supply of L liters of pesticide per month. The costs of these human hours and pesticide are suck cost and should not be considered in this problem. Your goal is to make a decision on splitting your field to maximize your total profit in a month.

   (a) Suppose the selling price of plant i per m^2 is P_i. Formulate an LP that solves your problem.
   (b) Continue from Part (a). Suppose that once you have plant 5 and 7 in your field, you cannot have plant 2. Add new variables and constraints into the LP in Part (b) to form an IP that solves your problem.
   (c) Continue from Part (a) and ignore Part (b). Suppose that a supplier contacts you and offers pesticide at the price of S dollars per liter. Explain how should you decide whether to buy some from the supplier. Please note that in any case, you have those L liters for free.
   (d) Ignore Parts (a) to (c). Suppose the selling price of plant i per m^2 is $B_i\sqrt{x_i}$, where x_i is the area (in m^2) of planting plant i. Formulate an NLP that solves your problem.
   (e) For the LP, IP, and NLP above, determine whether they are convex programs. Briefly explain.

6. (20 points; 5 points each) A firm is trying to design an app's two versions: a free version and a full version. The quality levels of the free and full versions are q_1 and q_2, respectively; the price of the full version is p. Naturally, $q_1 \leq q_2$, and once the R&D fee cq_2^2 is paid to build the full version, getting the weaker free version does not incur any other cost. Given the two versions of apps, consumers will decide whether to buy the full version, just download the free version, or do nothing based on how they like this app. There are N consumers, each has a willingness-to-pay per unit quality θ that is uniformly distributed between 0 and b. Let the consumer whose willingness-to-pay is θ be a type-θ consumer. A type-θ consumer's utility is

$$u(\theta) = \begin{cases} \theta q_2 - p & \text{if she buys the full version} \\ \theta q_1 & \text{if she downloads the free version} \\ 0 & \text{if she does nothing} \end{cases}.$$

   Each consumer acts to maximize her utility. The firm chooses $q_1 \geq 0$, $q_2 \geq 0$, and $p \geq 0$ to maximize its expected profit. Please note that the only cost of the firm is the R&D cost for the full version; reproducing the app does not incur any cost.

   (a) Given the two versions of the app, determine the expected number of consumers who will be willing to buy the full version. Express your answer as a function of q_1, q_2, and p.
   (b) Formulate the firm's optimization problem as an NLP?
   (c) If q_1 and q_2 are fixed, is the firm's objective function concave in p? Why or why not?
   (d) Show that $q_1 = 0$ in any optimal solution. Then provide an economic interpretation.

7. (15 points) A restaurant owner must schedule 10 workers to serve customers in the 14 hours (8 am to 10 pm) of operations in a day. Let these hours be labeled as hour 1 (8 am to 9 am), hour 2 (9 am to 10 am), ..., and hour 14 (9 pm to 10 pm), respectively. The number of workers required in hour i is D_i. Each worker should work for 8 hours a day (i.e., her work hours should occupy exactly 8 out of the 14 time slots). It is known that $\sum_{i=1}^{14} D_i \leq 80$. Consider the following two problems separately.

   (a) (5 points) Ideally, a worker should not work in both hour 1 and hour 14. Formulate an IP that schedules the workers to the 14 hours so that the number of workers who work in the ideal way is maximized.
   (b) (10 points) Ideally, in a day a worker should work for two shifts, one with four hours. Between the two shifts there should be at least one hour of break. Please formulate an IP that schedules the workers to the 14 hours so that the number of workers who work in the ideal way is maximized.