# Operations Research, Spring 2015
# Suggested Solution for Homework 1

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1. (a) Let the parameters be

$$D_i = \text{the demands for air conditioners of month } i,\ i = 1, ..., 6.$$

Let the decision variables be

$$h_j = \text{production quantity of month } i \text{ in Hsinchu},\ i = 1, ..., 6,$$
$$t_j = \text{production quantity of month } i \text{ in Taoyuan},\ i = 1, ..., 6,$$
$$x_i = \text{ending inventory of month } i,\ i = 1, ..., 6.$$

$$\begin{aligned}
\min \quad & \sum_{i=1}^{6}(400h_i + 350t_i + 80x_i) \\
\text{s.t.} \quad & 2000 + h_1 + t_1 - 2500 = x_1 \\
& x_{i-1} + h_i + t_i - Di = x_i \quad \forall i = 2, ..., 6 \\
& 2h_i \leq 4000 \quad \forall i = 1, ..., 6 \\
& 2.5t_i \leq 4000 \quad \forall i = 1, ..., 6. \\
& x_i \geq 0, \quad h_i \geq 0, \quad t_i \geq 0 \quad \forall i = 1, ..., 6
\end{aligned}$$

(b) Let the parameters be

$$D_i = \text{the maximum demands for air conditioners of month } i,\ i = 1, ..., 6.$$

Let the decision variables be

$$s_i = \text{sales quantity of month } i,\ i = 1, ..., 6,$$
$$h_i = \text{production quantity of month } i \text{ in Hsinchu},\ i = 1, ..., 6,$$
$$t_i = \text{production quantity of month } i \text{ in Taoyuan},\ i = 1, ..., 6,$$
$$x_i = \text{ending inventory of month } i,\ i = 1, ..., 6.$$

$$\begin{aligned}
\max \quad & \sum_{i=1}^{6}(600s_i - 400h_i - 350t_i - 80x_i) \\
\text{s.t.} \quad & 2000 + h_1 + t_1 - s_1 = x_1 \\
& x_1 + h_2 + t_2 - s_2 = x_2 \\
& x_2 + h_3 + t_3 - s_3 = x_3 \\
& x_3 + h_4 + t_4 - s_4 = x_4 \\
& x_4 + h_5 + t_5 - s_5 = x_5 \\
& x_5 + h_6 + t_6 - s_6 = x_6 \\
& x_i \geq 0 \quad \forall i = 1, ..., 6 \\
& s_i \leq D_i \quad \forall i = 1, ..., 6 \\
& 2h_i \leq 4000 \quad \forall i = 1, ..., 6 \\
& 2.5t_i \leq 4000 \quad \forall i = 1, ..., 6.
\end{aligned}$$

2. Let the parameters be

$$C_{ij} = \text{the cost for worker } i \text{ to 100\% complete job } j,\ i = 1, ..., m,\ j = 1, ..., n.$$

Let the decision variables be

$$a_{ij} = \text{the proportion of job } j \text{ that worker } i \text{ completes}, \; i = 1, ..., m, \; j = 1, ..., n.$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}C_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{m} a_{ij} = 1 \quad \forall j = 1, ..., n \\ & \sum_{j=1}^{n} a_{ij} \leq 2 \quad \forall i = 1, ..., m \\ & a_{ij} \in [0, 1] \quad \forall i = 1, ..., m, \; j = 1, ..., n. \end{aligned}$$

3. (a) The feasible region and isoquant line are illustrated in Figure 1. It is clear that we should push the isoquant line until we stop at the extreme point $(0, 9)$, which is an optimal solution.

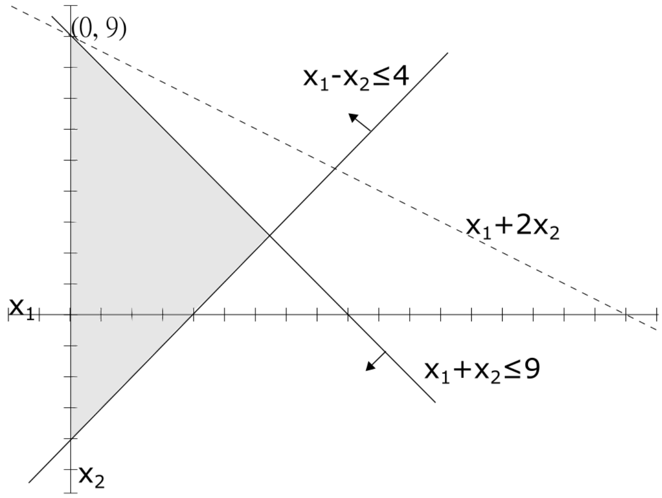


Figure 1: Graphical solution for Problem 3

(b) The standard form is

$$\begin{aligned} \max \quad & x_1 + 2x_2 - 2x_3 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 + x_4 = 4 \\ & x_1 + x_2 - x_3 + x_5 = 9 \\ & x_i \geq 0 \quad \forall i = 1, ..., 5. \end{aligned}$$

Since in the standard form we have five variables and three constraints, there should be three basic variables and two nonbasic variables in a basic solution. The ten possible ways to choose two (nonbasic) variables to be 0 are listed in the table below.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | basis | Basic feasible solution? |
|---|---|---|---|---|---|---|
| $\frac{13}{2}$ | $\frac{5}{2}$ | 0 | 0 | 0 | $\{x_1, x_2\}$ | Yes |
| $\frac{13}{2}$ | 0 | -$\frac{5}{2}$ | 0 | 0 | $\{x_1, x_3\}$ | No |
| 9 | 0 | 0 | -5 | 0 | $\{x_1, x_4\}$ | No |
| 4 | 0 | 0 | 0 | 5 | $\{x_1, x_5\}$ | Yes |
| 0 | – | – | 0 | 0 | $\{x_2, x_3\}$ | No |
| 0 | 9 | 0 | 13 | 0 | $\{x_2, x_4\}$ | Yes |
| 0 | -4 | 0 | 0 | 13 | $\{x_2, x_5\}$ | No |
| 0 | 0 | -9 | 13 | 0 | $\{x_3, x_4\}$ | No |
| 0 | 0 | 4 | 0 | 13 | $\{x_3, x_5\}$ | Yes |
| 0 | 0 | 0 | 4 | 9 | $\{x_4, x_5\}$ | Yes |

(c) The initial tableau is

$$\begin{array}{ccccc|c} -1 & -2 & 2 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 1 & 0 & x_4 = 4 \\ 1 & 1 & -1 & 0 & 1 & x_5 = 9 \end{array}$$

We use smallest index rule and run four iterations to get

$$\begin{array}{ccccc|c} -1 & -2 & 2 & 0 & 0 & 0 \\ \hline \boxed{1} & -1 & 1 & 1 & 0 & x_4 = 4 \\ 1 & 1 & -1 & 0 & 1 & x_5 = 9 \end{array} \rightarrow \begin{array}{ccccc|c} 0 & -3 & 3 & 1 & 0 & 4 \\ \hline 1 & -1 & 1 & 1 & 0 & x_1 = 4 \\ 0 & \boxed{2} & -2 & -1 & 1 & x_5 = 5 \end{array}$$

$$\rightarrow \begin{array}{ccccc|c} 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & \frac{23}{2} \\ \hline 1 & 0 & 0 & \boxed{\frac{1}{2}} & \frac{1}{2} & x_1 = \frac{13}{2} \\ 0 & 1 & -1 & -\frac{1}{2} & \frac{1}{2} & x_2 = \frac{5}{2} \end{array} \rightarrow \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 2 & 18 \\ \hline 2 & 0 & 0 & 1 & 1 & x_4 = 13 \\ 1 & 1 & -1 & 0 & 1 & x_2 = 9 \end{array}$$

an optimal solution to the original LP is $(x_1^*, x_2^*) = (0, 9)$ with objective value $z^* = 18$.

(d) The original LP becomes

$$\begin{aligned} \max \quad & x_1 + 2x_2 - 2x_3 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 + x_4 = 4 \\ & 2x_1 - x_2 + x_3 + x_5 = 10 \\ & x_i \geq 0 \quad \forall i = 1, ..., 5. \end{aligned}$$

The initial tableau is

$$\begin{array}{ccccc|c} -1 & -2 & 2 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 1 & 0 & x_4 = 4 \\ 2 & -1 & 1 & 0 & 1 & x_5 = 10 \end{array}$$

We use smallest index rule and run four iterations.

| | | | | | |
|---|---|---|---|---|---|
| $-1$ | $-2$ | $2$ | $0$ | $0$ | $0$ |
| $\boxed{1}$ | $-1$ | $1$ | $1$ | $0$ | $x_4 = 4$ |
| $2$ | $-1$ | $1$ | $0$ | $1$ | $x_5 = 10$ |

$\rightarrow$

| | | | | | |
|---|---|---|---|---|---|
| $0$ | $-3$ | $3$ | $1$ | $0$ | $4$ |
| $1$ | $-1$ | $1$ | $1$ | $0$ | $x_1 = 4$ |
| $0$ | $\boxed{1}$ | $-1$ | $-2$ | $1$ | $x_5 = 2$ |

$\rightarrow$

| | | | | | |
|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $-5$ | $3$ | $10$ |
| $1$ | $0$ | $0$ | $-1$ | $1$ | $x_1 = 6$ |
| $0$ | $1$ | $-1$ | $-2$ | $1$ | $x_2 = 2$ |

We can notice that $x_4$ is the only variable with negative coefficient in 1st row while its coefficient are all negative in other rows. It means that the constraint is unbounded, so we can't find the objective value in this modified LP.

4. (a) The exterme points are listed as follows.

| $x_1$ | $x_2$ | $x_3$ |
|---|---|---|
| (0 , | 0 , | 3) |
| (0 , | 0 , | 9) |
| (0 , | 6 , | 3) |
| (4 , | 0 , | 3) |
| (4 , | 0 , | 5) |
| (5 , | 1 , | 3) |

(b) The standard form LP is

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 - x_2 + x_4 = 4 \\ & x_1 + x_2 + x_3 + x_5 = 9 \\ & x_3 - x_6 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 6. \end{aligned}$$

We need to use two-phase implementation.

i. The Phase-I standard form LP is

$$\begin{aligned} \min \quad & x_7 \\ \text{s.t.} \quad & x_1 - x_2 + x_4 = 4 \\ & x_1 + x_2 + x_3 + x_5 = 9 \\ & x_3 - x_6 + x_7 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 7. \end{aligned}$$

First, solve the Phase-I LP which tries to minimize $x_7$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-1$ | $0$ |
| $1$ | $-1$ | $0$ | $1$ | $0$ | $0$ | $0$ | $x_4 = 4$ |
| $1$ | $1$ | $1$ | $0$ | $1$ | $0$ | $0$ | $x_5 = 9$ |
| $0$ | $0$ | $1$ | $0$ | $0$ | $-1$ | $1$ | $x_7 = 3$ |

$\rightarrow$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $1$ | $0$ | $0$ | $-1$ | $0$ | $3$ |
| $1$ | $-1$ | $0$ | $1$ | $0$ | $0$ | $0$ | $x_4 = 4$ |
| $1$ | $1$ | $1$ | $0$ | $1$ | $0$ | $0$ | $x_5 = 9$ |
| $0$ | $0$ | $\boxed{1}$ | $0$ | $0$ | $-1$ | $1$ | $x_7 = 3$ |

$\rightarrow$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ |
| $1$ | $-1$ | $0$ | $1$ | $0$ | $0$ | $0$ | $x_4 = 4$ |
| $1$ | $1$ | $0$ | $0$ | $1$ | $1$ | $-1$ | $x_5 = 6$ |
| $0$ | $0$ | $1$ | $0$ | $0$ | $-1$ | $1$ | $x_3 = 3$ |

ii. Then, solve the Phase-II LP.We use smallest index rule and run four iterations to get

$$
\begin{array}{cccccc|c}
-1 & -2 & 0 & 0 & 0 & 0 & 0 \\
\hline
\boxed{1} & -1 & 0 & 1 & 0 & 0 & x_4 = 4 \\
1 & 1 & 0 & 0 & 1 & 1 & x_5 = 6 \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}
\rightarrow
\begin{array}{cccccc|c}
0 & -3 & 0 & 1 & 0 & 0 & 4 \\
\hline
1 & -1 & 0 & 1 & 0 & 0 & x_1 = 4 \\
0 & \boxed{2} & 0 & -1 & 1 & 1 & x_5 = 2 \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}
$$

$$
\rightarrow
\begin{array}{cccccc|c}
0 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & 7 \\
\hline
1 & 0 & 0 & \boxed{\frac{1}{2}} & \frac{1}{2} & \frac{1}{2} & x_1 = 5 \\
0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & x_2 = 1 \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}
\rightarrow
\begin{array}{cccccc|c}
1 & 0 & 0 & 1 & 2 & 2 & 12 \\
\hline
2 & 0 & 0 & 1 & 1 & 1 & x_4 = 10 \\
1 & 1 & 0 & 0 & 1 & 1 & x_2 = 6 \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}
$$

an optimal solution to the LP is $(x_1^*, x_2^*) = (0, 6)$ with objective value $z^* = 12$. There isn't any iteration that has no improvement.

5. The standard form is

$$
\begin{aligned}
\max \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & x_1 - x_2 + x_4 = 4 \\
& x_1 + x_2 + x_3 + x_5 = 7 \\
& x_3 - x_6 = 3 \\
& x_i \geq 0 \quad \forall i = 1, ..., 6.
\end{aligned}
$$

(a) The bases of the problem correspond to four same bfs as below, so it's a degenerate LP.

| basis | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $\{x_1, x_2, x_3\}$ | 4 | 0 | 3 | 0 | 0 | 0 |
| $\{x_1, x_3, x_4\}$ | 4 | 0 | 3 | 0 | 0 | 0 |
| $\{x_1, x_3, x_5\}$ | 4 | 0 | 3 | 0 | 0 | 0 |
| $\{x_1, x_3, x_6\}$ | 4 | 0 | 3 | 0 | 0 | 0 |

(b) The LP has no trivial bfs. We need to use two-phase implementation.

i. First, solve the Phase-I LP which tries to minimize $x_6$.

$$
\begin{array}{ccccccc|c}
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
\hline
1 & -1 & 0 & 1 & 0 & 0 & 0 & x_4 = 4 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & x_5 = 7 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & x_7 = 3
\end{array}
\rightarrow
\begin{array}{ccccccc|c}
0 & 0 & 1 & 0 & 0 & -1 & -1 & 3 \\
\hline
1 & -1 & 0 & 1 & 0 & 0 & 0 & x_4 = 4 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & x_5 = 7 \\
0 & 0 & \boxed{1} & 0 & 0 & -1 & 1 & x_7 = 3
\end{array}
$$

$$
\rightarrow
\begin{array}{ccccccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
1 & -1 & 0 & 1 & 0 & 0 & 0 & x_4 = 4 \\
1 & 1 & 0 & 0 & 1 & 1 & -1 & x_5 = 4 \\
0 & 0 & 1 & 0 & 0 & -1 & 1 & x_3 = 3
\end{array}
$$

ii. Then, solve the Phase-II LP. We use smallest index rule and run four iterations.

$$
\begin{array}{cccccc|c}
-1 & -2 & 0 & 0 & 0 & 0 & 0 \\
\hline
\boxed{1} & -1 & 0 & 1 & 0 & 0 & x_4 = 4 \\
1 & 1 & 0 & 0 & 1 & 1 & x_5 = 4 \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}
\rightarrow
\begin{array}{cccccc|c}
0 & -3 & 0 & 1 & 0 & 0 & \mathbf{4} \\
\hline
1 & -1 & 0 & 1 & 0 & 0 & \mathbf{x_1 = 4} \\
0 & \boxed{2} & 0 & -1 & 1 & 1 & \mathbf{x_5 = 0} \\
0 & 0 & 1 & 0 & 0 & -1 & \mathbf{x_3 = 3}
\end{array}
$$

$$
\rightarrow
\begin{array}{cccccc|c}
0 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & \frac{3}{2} & \mathbf{4} \\
\hline
1 & 0 & 0 & \boxed{\frac{1}{2}} & \frac{1}{2} & \frac{1}{2} & \mathbf{x_1 = 4} \\
0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \mathbf{x_2 = 0} \\
0 & 0 & 1 & 0 & 0 & -1 & \mathbf{x_3 = 3}
\end{array}
\rightarrow
\begin{array}{cccccc|c}
1 & 0 & 0 & 0 & 2 & 2 & 8 \\
\hline
2 & 0 & 0 & 1 & 1 & 1 & x_4 = 8 \\
1 & 1 & 0 & 0 & 1 & 1 & x_2 = 4 \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}
$$

The steps with highlight are degenerate basic feasible solutions, and they are also the iterations that has no improvement.

6. The standard form is

$$
\begin{array}{rl}
\max & x_1 + 2x_2 \\
\text{s.t.} & x_1 - x_2 + x_4 = 4 \\
& x_1 + x_2 + x_3 + x_5 = 9 \\
& x_3 - x_6 = 3 \\
& x_i \geq 0 \quad \forall i = 1, ..., 6.
\end{array}
$$

(a)

$$
A_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad A_N = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad c_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad c_N = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix}
$$

(b) The reduced costs are

$$
c_N^{-T} = c_B^T A_B^{-1} A_N - c_N^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 \end{bmatrix}
$$

$\rightarrow$ We choose $x_2$ to enter because its reduced cost is the most negative among the nonbasic varibales.
$\rightarrow$ $x_j = x_2$.

(c)

$$
A_B^{-1} b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}
$$

$$
A_B^{-1} A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}
$$

$$
\rightarrow \text{ratio test: } \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{4}{-1} \\ \frac{3}{0} \\ \frac{2}{2} \end{bmatrix} \rightarrow x_5 \text{ leaves.}
$$