Operations Research

The Simplex Method (Part 1)

Ling-Chieh Kung

Department of Information Management
National Taiwan University
Introduction

- In these two lectures, we will study how to solve an LP.
- The algorithm we will introduce is the **simplex method**.
  - Developed by George Dantzig in 1947.
  - Opened the whole field of Operations Research.
  - Implemented in most commercial LP solvers.
  - Very efficient for almost all practical LPs.
  - With very simple ideas.
- The method is general in an indirect manner.
  - There are many different forms of LPs.
  - We will first show that each LP is equivalent to a **standard form** LP.
  - Then we will show how to solve standard form LPs.
- Read Sections 4.1 to 4.4 of the textbook thoroughly!
- These two lectures will be full of algebra and theorems. Get ready!
Road map

- **Standard form LPs.**
- Basic solutions.
- Basic feasible solutions.
- The geometry of the simplex method.
- The algebra of the simplex method.
First, let’s define the **standard form**.¹

**Definition 1 (Standard form LP)**

*An LP is in the standard form if*

- all the RHS values are nonnegative,
- all the variables are nonnegative, and
- all the constraints are equalities.

- RHS = right hand sides. For any constraint

  \[ g(x) \leq b, \quad g(x) \geq b, \quad \text{or} \quad g(x) = b, \]

  \( b \) is the RHS.

- There is no restriction on the objective function.

¹In the textbook, this form is called the augmented form. In the world of OR, however, “standard form” is a more common name for LPs in this format.
Finding the standard form

- How to find the standard form for an LP?
- Requirement 1: **Nonnegative RHS.**
  - If it is negative, **switch** the LHS and the RHS.
  - E.g.,
    \[2x_1 + 3x_2 \leq -4\]
    is equivalent to
    \[-2x_1 - 3x_2 \geq 4.\]
Finding the standard form

- **Requirement 2: Nonnegative variables.**
  - If $x_i$ is nonpositive, replace it by $-x_i$. E.g.,
    \[
    2x_1 + 3x_2 \leq 4, \quad x_1 \leq 0 \quad \Leftrightarrow \quad -2x_1 + 3x_2 \leq 4, \quad x_1 \geq 0.
    \]
  - If $x_i$ is free, replace it by $x_i' - x_i''$, where $x_i', x_i'' \geq 0$. E.g.,
    \[
    2x_1 + 3x_2 \leq 4, \quad x_1 < 0 \quad \Leftrightarrow \quad 2x_1' - 2x_1'' + 3x_2 \leq 4, \quad x_1' \geq 0, \quad x_1'' \geq 0.
    \]

<table>
<thead>
<tr>
<th>$x_i = x_i' - x_i''$</th>
<th>$x_i' \geq 0$</th>
<th>$x_i'' \geq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>−8</td>
<td>0</td>
<td>8</td>
</tr>
</tbody>
</table>
Finding the standard form

- Requirement 3: **Equality constraints.**
  - For a “≤” constraint, add a slack variable. E.g.,
    \[2x_1 + 3x_2 \leq 4 \iff 2x_1 + 3x_2 + x_3 = 4, \quad x_3 \geq 0.\]
  - For a “≥” constraint, minus a surplus/excess variable. E.g.,
    \[2x_1 + 3x_2 \geq 4 \iff 2x_1 + 3x_2 - x_3 = 4, \quad x_3 \geq 0.\]
  - For ease of exposition, they will both be called slack variables.
  - A slack variable measures the gap between the LHS and RHS.
An example

\[ \begin{align*}
\text{min} & \quad 3x_1 + 2x_2 + 4x_3 \\
\text{s.t.} & \quad x_1 + 2x_2 - x_3 \geq 6 \\
& \quad x_1 - x_2 \geq -8 \\
& \quad 2x_1 + x_2 + x_3 = 9 \\
& \quad x_1 \geq 0, \ x_2 \leq 0, \ x_3 \text{ urs.}
\end{align*} \]

\[ \begin{align*}
\text{min} & \quad 3x_1 + 2x_2 + 4x_3 \\
\rightarrow & \quad \text{s.t.} \\
& \quad x_1 + 2x_2 - x_3 \geq 6 \\
& \quad -x_1 + x_2 \leq 8 \\
& \quad 2x_1 + x_2 + x_3 = 9 \\
& \quad x_1 \geq 0, \ x_2 \leq 0, \ x_3 \text{ urs.}
\end{align*} \]
An example

\[
\begin{align*}
\text{min} & \quad 3x_1 - 2x_2 + 4x_3 - 4x_4 \\
\rightarrow \text{s.t.} & \quad x_1 - 2x_2 - x_3 + x_4 \geq 6 \\
& \quad -x_1 - x_2 \leq 8 \\
& \quad 2x_1 - x_2 + x_3 - x_4 = 9 \\
& \quad x_i \geq 0 \quad \forall i = 1, ..., 4
\end{align*}
\]

\[
\begin{align*}
\text{min} & \quad 3x_1 - 2x_2 + 4x_3 - 4x_4 \\
\rightarrow \text{s.t.} & \quad x_1 - 2x_2 - x_3 + x_4 - x_5 = 6 \\
& \quad -x_1 - x_2 + x_6 = 8 \\
& \quad 2x_1 - x_2 + x_3 - x_4 = 9 \\
& \quad x_i \geq 0 \quad \forall i = 1, ..., 6.
\end{align*}
\]
Standard form LPs in matrices

- Given any LP, we may find its standard form.
- With matrices, a standard form LP is expressed as

\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

- E.g., for

\[
\begin{align*}
\text{min} & \quad 2x_1 - x_2 \\
\text{s.t.} & \quad x_1 + 5x_2 + x_3 = 5 \\
& \quad 3x_1 - 6x_2 + x_4 = 4 \\
& \quad x_i \geq 0 \quad \forall i = 1, \ldots, 4,
\end{align*}
\]

\[
c = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \text{and}
\]

\[
A = \begin{bmatrix} 1 & 5 & 1 & 0 \\ 3 & -6 & 0 & 1 \end{bmatrix}.
\]

- We will denote the number of constraints and variables as \( m \) and \( n \).
  - \( A \in \mathbb{R}^{m \times n} \) is called the coefficient matrix.
  - \( b \in \mathbb{R}^m \) is called the RHS vector.
  - \( c \in \mathbb{R}^n \) is called the objective vector.
- The objective function can be either max or min.
Solving standard form LPs

- So now we only need to find a way to solve standard form LPs.
- How?
- A standard form LP is still an LP.
- If it has an optimal solution, it has an extreme point optimal solution! Therefore, we only need to search among extreme points.
- Our next step is to understand more about the extreme points of a standard form LP.
Road map

- Standard form LPs.
- **Basic solutions**.
- Basic feasible solutions.
- The geometry of the simplex method.
- The algebra of the simplex method.
Bases

- Consider a standard form LP with $m$ constraints and $n$ variables

$$
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
$$

- We may assume that rank $A = m$, i.e., all rows of $A$ are independent.\(^2\)
- This then implies that $m \leq n$. As the problem with $m = n$ is trivial, we will assume that $m < n$.

- For the system $Ax = b$, now there are more columns than rows. Let’s select some columns to form a basis:

**Definition 2 (Basis)**

\[ A \text{ basis } B \text{ of a standard form LP is a selection of } m \text{ variables such that } A_B, \text{ the matrix formed by the } m \text{ corresponding columns of } A, \text{ is invertible/nonsingular.} \]

\(^2\)This assumption is without loss of generality. Why?
Basic solutions

- By ignoring the other $n - m$ variables, $Ax = b$ will have a unique solution (because $A_B$ is invertible).
- Each basis uniquely defines a **basic solution**:

<table>
<thead>
<tr>
<th>Definition 3 (Basic solution)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A basic solution to a standard form LP is a solution that (1) has $n - m$ variables being equal to 0 and (2) satisfies $Ax = b$.</td>
</tr>
</tbody>
</table>

- The $n - m$ variables chosen to be zero are **nonbasic variables**.
- The remaining $m$ variables are **basic variables**. They form a basis (i.e., $A_B^{-1}$ is invertible; otherwise $Ax = b$ has no solution).
- We use $x_B \in \mathbb{R}^m$ and $x_N \in \mathbb{R}^{n-m}$ to denote basic and nonbasic variables, respectively, with respect to a given basis $B$.
  - We have $x_N = 0$ and $x_B = A_B^{-1}b$.
  - Note that a basic variable may be positive, negative, or zero!
Basic solutions: an example

- Consider an original LP

\[
\begin{align*}
\text{min} & \quad 6x_1 + 8x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 \leq 6 \\
& \quad 2x_1 + x_2 \leq 6 \\
& \quad x_i \geq 0 \quad \forall i = 1,2
\end{align*}
\]

and its standard form

\[
\begin{align*}
\text{min} & \quad 6x_1 + 8x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 + x_3 = 6 \\
& \quad 2x_1 + x_2 + x_4 = 6 \\
& \quad x_i \geq 0 \quad \forall i = 1, \ldots, 4.
\end{align*}
\]
Basic solutions: an example

- In the standard form, $m = 2$ and $n = 4$.
  - There are $n - m = 2$ nonbasic variables.
  - There are $m = 2$ basic variables.
- Steps for obtaining a basic solution:
  - Determine a set of $m$ basic variables to form a basis $B$.
  - The remaining variables form the set of nonbasic variables $N$.
  - Set nonbasic variables to zero: $x_N = 0$.
  - Solve the $m$ by $m$ system $A_B x_B = b$ for the values of basic variables.
- For this example, we will solve a two by two system for each basis.
Basic solutions: an example

- The two equalities are

\[ \begin{align*}
    x_1 + 2x_2 + x_3 &= 6 \\
    2x_1 + x_2 + x_4 &= 6.
\end{align*} \]

- Let’s try \( B = \{x_1, x_2\} \) and \( N = \{x_3, x_4\} \):

\[ \begin{align*}
    x_1 + 2x_2 &= 6 \\
    2x_1 + x_2 &= 6.
\end{align*} \]

The solution is \((x_1, x_2) = (2, 2)\). Therefore, the basic solution associated with this basis \( B \) is \((x_1, x_2, x_3, x_4) = (2, 2, 0, 0)\).

- Let’s try \( B = \{x_2, x_3\} \) and \( N = \{x_1, x_4\} \):

\[ \begin{align*}
    2x_2 + x_3 &= 6 \\
    x_2 &= 6.
\end{align*} \]

As \((x_2, x_3) = (6, -6)\), the basic solution is \((x_1, x_2, x_3, x_4) = (0, 6, -6, 0)\).
Bases

- In general, as we need to choose $m$ out of $n$ variables to be basic, we have at most $\binom{n}{m}$ different bases.$^3$

- In this example, we have exactly $\binom{4}{2} = 6$ bases.

- By examining all the six bases one by one, we may find all those associated basic variables:

<table>
<thead>
<tr>
<th>Basis</th>
<th>Basic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_1$</td>
</tr>
<tr>
<td>${x_1, x_2}$</td>
<td>2</td>
</tr>
<tr>
<td>${x_1, x_3}$</td>
<td>3</td>
</tr>
<tr>
<td>${x_1, x_4}$</td>
<td>6</td>
</tr>
<tr>
<td>${x_2, x_3}$</td>
<td>0</td>
</tr>
<tr>
<td>${x_2, x_4}$</td>
<td>0</td>
</tr>
<tr>
<td>${x_3, x_4}$</td>
<td>0</td>
</tr>
</tbody>
</table>

$^3$Why “at most”? Why not “exactly”?

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Basic solutions v.s. bases

- For a basis, what matters are **variables**, not **values**.
- Consider another example

\[
\begin{align*}
\text{min} & \quad 6x_1 + 8x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 \leq 6 \\
& \quad 2x_1 + x_2 \leq 12 \\
& \quad x_i \geq 0 \quad \forall i = 1, 2
\end{align*}
\]

and its standard form

\[
\begin{align*}
\text{min} & \quad 6x_1 + 8x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 + x_3 = 6 \\
& \quad 2x_1 + x_2 + x_4 = 12 \\
& \quad x_i \geq 0 \quad \forall i = 1, \ldots, 4.
\end{align*}
\]
Basic solutions v.s. bases

- The six bases and the associated basic variables are listed below:

<table>
<thead>
<tr>
<th>Basis</th>
<th>Basic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x_1$</td>
</tr>
<tr>
<td>${x_1, x_2}$</td>
<td>6</td>
</tr>
<tr>
<td>${x_1, x_3}$</td>
<td>6</td>
</tr>
<tr>
<td>${x_1, x_4}$</td>
<td>6</td>
</tr>
<tr>
<td>${x_2, x_3}$</td>
<td>0</td>
</tr>
<tr>
<td>${x_2, x_4}$</td>
<td>0</td>
</tr>
<tr>
<td>${x_3, x_4}$</td>
<td>0</td>
</tr>
</tbody>
</table>

- Three different bases result in the same basic solution!
- There are six distinct bases but only four distinct basic solutions.
- Number of distinct basic solutions $\leq$ number of distinct bases $\leq \binom{n}{m}$.
- When multiple bases correspond to one single basic solution, the LP is degenerate; otherwise, it is nondegenerate.
- We will discuss degeneracy only at the end of the next lecture.
Road map

- Standard form LPs.
- Basic solutions.
- **Basic feasible solutions.**
- The geometry of the simplex method.
- The algebra of the simplex method.
Basic feasible solutions

Among all basic solutions, some are feasible.

- By the definition of basic solutions, they satisfy $Ax = b$.
- If one also satisfies $x \geq 0$, it satisfies all constraints.

In this case, it is called **basic feasible solutions** (bfs).\(^4\)

**Definition 4 (Basic feasible solution)**

A basic feasible solution to a standard form LP is a basic solution whose basic variables are all nonnegative.

<table>
<thead>
<tr>
<th>Basis</th>
<th>Basic solution</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x_1, x_2}$</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>${x_1, x_3}$</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>${x_1, x_4}$</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$-6$</td>
<td></td>
</tr>
<tr>
<td>${x_2, x_3}$</td>
<td>0</td>
<td>6</td>
<td>$-6$</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>${x_2, x_4}$</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>${x_3, x_4}$</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

\(^4\)In the textbook, the abbreviation is “BF solutions”.

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Basic feasible solutions and extreme points

- Why bfs are important?
- They are just extreme points!

**Proposition 1 (Extreme points and basic feasible solutions)**

For a standard form LP, a solution is an extreme point of the feasible region if and only if it is a basic feasible solution to the LP.

*Proof.* Beyond the scope of this course.

- Though we cannot prove it here, let’s get some intuitions with graphs.\(^5\)

---

\(^5\)Please note that these “intuitions” are never rigorous.
An example

- There is a one-to-one mapping between bfs and extreme points.

<table>
<thead>
<tr>
<th>Basis</th>
<th>Bfs?</th>
<th>Point</th>
<th>Basic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$x_1$</td>
</tr>
<tr>
<td>${x_1, x_2}$</td>
<td>Yes</td>
<td>$A$</td>
<td>2</td>
</tr>
<tr>
<td>${x_1, x_3}$</td>
<td>Yes</td>
<td>$B$</td>
<td>3</td>
</tr>
<tr>
<td>${x_1, x_4}$</td>
<td>No</td>
<td>$C$</td>
<td>6</td>
</tr>
<tr>
<td>${x_2, x_3}$</td>
<td>No</td>
<td>$D$</td>
<td>0</td>
</tr>
<tr>
<td>${x_2, x_4}$</td>
<td>Yes</td>
<td>$E$</td>
<td>0</td>
</tr>
<tr>
<td>${x_3, x_4}$</td>
<td>Yes</td>
<td>$F$</td>
<td>0</td>
</tr>
</tbody>
</table>
Another example

Would you find the one-to-one correspondence?

<table>
<thead>
<tr>
<th>Basis</th>
<th>Basic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>{x_1, x_2}</td>
<td>\begin{align*} x_1 &amp;= 6 \ x_2 &amp;= 0 \ x_3 &amp;= 0 \ x_4 &amp;= 0 \end{align*}</td>
</tr>
<tr>
<td>{x_1, x_3}</td>
<td>\begin{align*} x_1 &amp;= 6 \ x_2 &amp;= 0 \ x_3 &amp;= 0 \ x_4 &amp;= 0 \end{align*}</td>
</tr>
<tr>
<td>{x_1, x_4}</td>
<td>\begin{align*} x_1 &amp;= 6 \ x_2 &amp;= 0 \ x_3 &amp;= 0 \ x_4 &amp;= 0 \end{align*}</td>
</tr>
<tr>
<td>{x_2, x_3}</td>
<td>\begin{align*} x_1 &amp;= 0 \ x_2 &amp;= 12 \ x_3 &amp;= -18 \ x_4 &amp;= 0 \end{align*}</td>
</tr>
<tr>
<td>{x_2, x_4}</td>
<td>\begin{align*} x_1 &amp;= 0 \ x_2 &amp;= 3 \ x_3 &amp;= 0 \ x_4 &amp;= 9 \end{align*}</td>
</tr>
<tr>
<td>{x_3, x_4}</td>
<td>\begin{align*} x_1 &amp;= 0 \ x_2 &amp;= 0 \ x_3 &amp;= 6 \ x_4 &amp;= 12 \end{align*}</td>
</tr>
</tbody>
</table>
Optimality of basic feasible solutions

- What’s the implication of the previous proposition?

**Proposition 2 (Optimality of basic feasible solutions)**

*For a standard form LP, if there is an optimal solution, there is an optimal basic feasible solution.*

**Proof.** We know if there is an optimal solution, there is an optimal extreme point solution. Moreover, we know extreme points are just bfs. The proof then follows.
Solving standard form LPs

- To find an optimal solution:
  - Instead of searching among all extreme points, we may search among all bfs.
- But the two sets are equally large! What is the difference?
  - Extreme points are defined with geometry but bfs are with algebra.
  - Checking whether a solution is an extreme point is hard (for a computer).
  - Checking whether a solution is basic feasible is easy (for a computer).
- Given an LP:
  - Enumerating all extreme points is hard.
  - Enumerating all bfs is possible.
Solving standard form LPs

- We are now closer to solve a general LP:
  - We may enumerate all the bfs, compare them, and find the best one.
  - If this LP has an optimal solution, that best bfs is optimal.

- Unfortunately:
  - For a standard form LP with $n$ variables and $m$ constraints, we have at most $\binom{n}{m}$ bfs. Listing them takes too much time!\(^6\)

- We need to improve the search procedure.
  - We need to analyze bfs more deeply.
  - We need to understand how they are connected.

- Let’s define adjacent bfs.

\(^6\)The complexity is $O\left(\binom{n}{m}\right) = O(n!)$; it is an exponential-time algorithm.
Adjacent basic feasible solutions

Two bfs are either adjacent or not:

Definition 5 (Adjacent bases and bfs)

Two bases are adjacent if exactly one of their variable is different.
Two bfs are adjacent if their associated bases are adjacent.

- \( \{x_1, x_2\} \) and \( \{x_1, x_4\} \) are adjacent.
- \( \{x_1, x_2\} \) and \( \{x_3, x_4\} \) are not adjacent.
- How about \( \{x_1, x_2\} \) and \( \{x_2, x_4\} \)?
Adjacent basic feasible solutions

- A pair of adjacent bfs corresponds to a pair of “adjacent” extreme points, i.e., extreme points that are on the same edge.
- Switching from a bfs to its adjacent bfs is moving along an edge.

<table>
<thead>
<tr>
<th>Basis</th>
<th>Point</th>
<th>Basic solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>{x_1, x_2}</td>
<td>A</td>
<td>\begin{align*} x_1 &amp; = 2 \ x_2 &amp; = 2 \ x_3 &amp; = 0 \ x_4 &amp; = 0 \end{align*}</td>
</tr>
<tr>
<td>{x_1, x_3}</td>
<td>B</td>
<td>\begin{align*} x_1 &amp; = 3 \ x_2 &amp; = 0 \ x_3 &amp; = 3 \ x_4 &amp; = 0 \end{align*}</td>
</tr>
<tr>
<td>{x_2, x_4}</td>
<td>E</td>
<td>\begin{align*} x_1 &amp; = 0 \ x_2 &amp; = 3 \ x_3 &amp; = 0 \ x_4 &amp; = 3 \end{align*}</td>
</tr>
<tr>
<td>{x_3, x_4}</td>
<td>F</td>
<td>\begin{align*} x_1 &amp; = 0 \ x_2 &amp; = 0 \ x_3 &amp; = 6 \ x_4 &amp; = 6 \end{align*}</td>
</tr>
</tbody>
</table>
A three-dimensional example

\[ \begin{align*}
\text{min } & \quad \text{whatever} \\
\text{s.t. } & \quad x_1 + x_3 + x_4 \\
& \quad x_2 \\
& \quad x_3 \\
& \quad x_i \geq 0 \quad \forall i = 1, \ldots, 6.
\end{align*} \]

\[
\begin{array}{c|ccc|c|cc}
\text{Basis} & \text{Point} & \text{Basic solution} \\
& & x_1 & x_2 & x_3 \\
\hline
\{x_4, x_5, x_6\} & A & 0 & 0 & 0 \\
\{x_1, x_5, x_6\} & B & 2 & 0 & 0 \\
\{x_1, x_5, x_6\} & C & 2 & 1 & 0 \\
\{x_2, x_4, x_6\} & D & 0 & 1 & 0 \\
\{x_3, x_4, x_5\} & E & 0 & 0 & 1 \\
\{x_1, x_3, x_5\} & F & 1 & 0 & 1 \\
\{x_1, x_2, x_3\} & G & 1 & 1 & 1 \\
\{x_2, x_3, x_4\} & H & 0 & 1 & 1 \\
\end{array}
\]
A better way to search

- Given all these concepts, how would you search among bfs?
- At each bfs, move to an adjacent bfs that is better!
  - Around the current bfs, there should be some improving directions.
  - Otherwise, the bfs is optimal.
- Next we will introduce the simplex method, which utilize this idea in an elegant way.
Road map

- Standard form LPs.
- Basic solutions.
- Basic feasible solutions.
- The geometry of the simplex method.
- The algebra of the simplex method.
The simplex method

- All we need is to search among bfs.
  - Geometrically, we search among extreme points.
  - Moving to an adjacent bfs is to move along an edge.

- Questions:
  - Which edge to move along?
  - When to stop moving?

- All these must be done with algebra rather than geometry.
  - Nevertheless, geometry provides intuitions.

- Algebraically, to move to an adjacent bfs, we need to replace one basic variable by a nonbasic variable.
  - E.g., moving from $B_1 = \{x_1, x_2, x_3\}$ to $B_2 = \{x_2, x_3, x_5\}$.

- There are two things to do:
  - Select one nonbasic variable to enter the basis, and
  - Select one basic variable to leave the basis.
The entering variable

- Selecting one nonbasic variable to enter means making it **nonzero**.
- One constraint becomes **nonbinding**.
- We move along the edge that moves **away from** the constraint.
- We will illustrate this idea with the following LP

\[
\begin{align*}
\text{min} & \quad -x_1 \\
\text{s.t.} & \quad 2x_1 - x_2 \leq 4 \\
& \quad 2x_1 + x_2 \leq 8 \\
& \quad x_2 \leq 3 \\
& \quad x_i \geq 0 \quad \forall \ i = 1, 2.
\end{align*}
\]

and its standard form

\[
\begin{align*}
\text{min} & \quad -x_1 \\
\text{s.t.} & \quad 2x_1 - x_2 + x_3 = 4 \\
& \quad 2x_1 + x_2 + x_4 = 8 \\
& \quad x_2 + x_5 = 3 \\
& \quad x_i \geq 0 \quad \forall \ i = 1, \ldots, 5.
\end{align*}
\]
The entering variable

- For the bfs $x^1 = (0, 0, 4, 8, 3)$:
  - The basis is $\{x_3, x_4, x_5\}$.
  - $x_1$ and $x_2$ are nonbasic.
  - $x_1$ and $x_2$ may enter the basis.
  - Letting $x_1$ enters
    - making $x_1 > 0$
    - moving away from $x_1 \geq 0$
    - moving along direction $A$.
  - Letting $x_2$ enters
    - making $x_2 > 0$
    - moving away from $x_2 \geq 0$
    - moving along direction $B$. 

The diagram illustrates the feasible region defined by the constraints $2x_1 + x_2 \leq 8$, $x_2 \leq 3$, $2x_1 - x_2 \leq 4$, and $x_1 \geq 0$, $x_2 \geq 0$. The starting basic feasible solution $x^1$ is shown at the origin, and the feasible region is shaded. The direction $A$ corresponds to the entry of $x_1$, and the direction $B$ corresponds to the entry of $x_2$. The diagram also includes the vertices $x^1$, $B$, $A$, and $x_5$.
The entering variable

- For the bfs $x^2 = (2, 0, 0, 4, 3)$:
  - The basis is $\{x_1, x_4, x_5\}$.
  - $x_2$ and $x_3$ are nonbasic.
  - $x_2$ and $x_3$ may enter the basis.
  - Letting $x_2$ enters
    - making $x_2 > 0$
    - moving away from $x_2 \geq 0$
    - moving along direction $D$.
  - Letting $x_3$ enters
    - making $x_3 > 0$
    - moving away from $2x_1 - x_2 + x_3 = 4$
    - moving along direction $C$. 
  
![Graph showing the simplex method process](image)
The leaving variable

- Suppose we have chosen one entering variable.
  - We have chosen one edge to move along.
- How to choose a **leaving** variable?
  - When should we **stop**?
- Geometrically, we stop when we “**hit a constraint**”.
  - We are moving along edges, so all equalities constraints will remain to be satisfied. Only nonnegativity constraints may be violated.
- Algebraically, we stop when one basic variable **decreases to 0**.
  - This basic variable will leave the basis.
  - As it becomes 0, it becomes a nonbasic variable.
The leaving variable

For the bfs \( x^1 \), suppose we move along direction \( A \).

- The original basis is \( \{x_3, x_4, x_5\} \).
- \( x_1 \) enters the basis.
- We first hit \( 2x_1 - x_2 \leq 4 \).
- \( x_3 \) becomes 0.
- \( x_3 \) becomes nonbasic.
- \( x_3 \) leaves the basis.
- The new basis is \( \{x_1, x_4, x_5\} \).
The leaving variable

- For the bfs $x^2$, suppose we move along direction $D$.
  - The original basis is $\{x_1, x_4, x_5\}$.
  - $x_2$ enters the basis.
  - We first hit $2x_1 + x_2 \leq 8$.
  - $x_4$ becomes 0.
  - $x_4$ becomes nonbasic.
  - $x_4$ leaves the basis.
  - The new basis is $\{x_1, x_2, x_5\}$.
An iteration

- At a bfs, we move to another **better** bfs.
  - We first choose **which direction to go** (the **entering** variable). That should be an improving direction along an edge.
  - We then determine **when to stop** (the **leaving** variable). That depends on the first constraint we hit.
  - We may then treat the new bfs as the current bfs and then **repeat**.
- We stop when there is no improving direction.
- The process of moving to the next bfs is call an **iteration**.
The simplex method

- The simplex method is simple:
  - It suffices to move along edges (because we only need to search among extreme points).
  - At each point, the number of directions to search for is small (because we consider only edges).
  - For each improving direction, the stopping condition is simple: Keep moving forwards until we cannot.

- The simplex method is smart:
  - When at a point there is no improving direction along an edge, the point is optimal.

- Next let’s know exactly how to run the simplex method in algebra.
Road map

- Standard form LPs.
- Basic solutions.
- Basic feasible solutions.
- The geometry of the simplex method.
- The algebra of the simplex method.
The simplex method

To introduce the algebra of the simplex method, let’s consider the following LP

\[
\begin{align*}
\text{min} & \quad -2x_1 - 3x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 \leq 6 \\
& \quad 2x_1 + x_2 \leq 8 \\
& \quad x_i \geq 0 \quad \forall \ i = 1, 2
\end{align*}
\]

and its standard form

\[
\begin{align*}
\text{min} & \quad -2x_1 - 3x_2 \\
\text{s.t.} & \quad x_1 + 2x_2 + x_3 = 6 \\
& \quad 2x_1 + x_2 + x_4 = 8 \\
& \quad x_i \geq 0 \quad \forall \ i = 1, ..., 4.
\end{align*}
\]
System of equalities

- We need to keep track of the \textit{objective value}.
  - We want to keep improving our solution.
  - We will use \( z = -2x_1 - 3x_2 \) to denote the objective value.
  - The objective value will sometimes be called \textit{the z value}.

- Once we keep in mind that (1) we are minimizing \( z \) and (2) all variables (except \( z \)) must be nonnegative, the standard form is nothing but a system of three equalities:
  \[
  \begin{align*}
  z + 2x_1 + 3x_2 & = 0 \\
  x_1 + 2x_2 + x_3 & = 6 \\
  2x_1 + x_2 + x_4 & = 8.
  \end{align*}
  \]

  - Note that \( z = -2x_1 - 3x_2 \) is expressed as \( z + 2x_1 + 3x_2 = 0 \).
  - This “constraint” (which actually represents the objective function) will be called the 0th constraint.

- We will repeatedly use Linear Algebra to solve the system.
An initial bfs

- To start, we need to first have an initial bfs.
  - For this example, a basis is a set of two variables such that $A_B$, the matrix formed by the two corresponding columns, is invertible.
  - By satisfying $A_Bx_B = b$, a bfs has all its basic variables $x_B$ nonnegative.
  - How may we get one bfs?

- Investigate the system in details:

$$
egin{align*}
  z &+ 2x_1 + 3x_2 & = 0 \\
  x_1 + 2x_2 + x_3 & = 6 \\
  2x_1 + x_2 + x_4 & = 8.
\end{align*}
$$

- Selecting $x_3$ and $x_4$ definitely works!
- In the system, these two columns form an identity matrix: $A_B = I$.\(^7\)
- Moreover, in a standard form LP, the RHS $b$ are nonnegative.
- Therefore, $x_B = A_B^{-1}b = Ib = b \geq 0$.

---

\(^7\)For what kind of LPs does this identity matrix exist?
Improving the current bfs

\[
\begin{align*}
    z + 2x_1 + 3x_2 &= 0 \\
    x_1 + 2x_2 + x_3 &= 6 \\
    2x_1 + x_2 + x_4 &= 8.
\end{align*}
\]

▶ Let us start from \(x^1 = (0, 0, 6, 8)\) and \(z_1 = 0\).
▶ To move, let’s choose a nonbasic variable to enter. \(x_1\) or \(x_2\)?
  ▶ The 0th constraints tells us that entering either variable makes \(z\) smaller: When one goes up, \(z\) goes down to maintain the equality.
  ▶ For no reason, let’s choose \(x_1\) to enter.
▶ When to stop?
  ▶ Now \(x_1\) goes up from 0.
  ▶ \((0, 0, 6, 8) \rightarrow (1, 0, 5, 6) \rightarrow (2, 0, 4, 4) \rightarrow \cdots\). Note that \(x_2\) remains 0.
  ▶ We will stop at \((4, 0, 2, 0)\), i.e., when \(x_4\) becomes 0.
  ▶ This is indicated by the ratio of the RHS and entering column: Because \(\frac{8}{2} < \frac{6}{1}\), \(x_4\) becomes 0 sooner than \(x_3\).
▶ We move to \(x^2 = (4, 0, 2, 0)\) with \(z_2 = -8\).
Keep improving the current bfs

\[
\begin{align*}
    z & + 2x_1 + 3x_2 = 0 \\
    x_1 + 2x_2 + x_3 & = 6 \\
    2x_1 + x_2 + x_4 & = 8.
\end{align*}
\]

▷ So far so good!
▷ Let’s improve \( x^2 = (4, 0, 2, 0) \) by moving to the next bfs.
   ▷ One of \( x_2 \) and \( x_4 \) may enter.
▷ According to the 0th row, we should let \( x_2 \) enter.\(^8\)
▷ When \( x_2 \) goes up and \( x_4 \) remains 0:
   ▷ The 2nd row says \( x_2 \) can at most become 8 (and then \( x_1 \) becomes 0).
   ▷ In the 1st row... how will \( x_1 \) and \( x_3 \) change???????
▷ An easier way is to update the system before the 2nd move.
   ▷ So that in each row there is only one basic variable.
▷ Let’s see how to update the system every time when we make a move.

\(^8\)This statement is actually wrong. Why?
Rewriting the standard form

- Recall that a standard form LP is

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0.
\end{align*}
\]

- Given a basis \( B \), we may split \( x \) into \((x_B, x_N)\).
- We may also split \( c \) into \((c_B, c_N)\) and \( A \) into \((A_B, A_N)\).
  - \( c_B \in \mathbb{R}^m \), \( c_N \in \mathbb{R}^{n-m} \), \( A_B \in \mathbb{R}^{m \times m} \), and \( A_N \in \mathbb{R}^{m \times (n-m)} \).
- With the splits, the LP becomes

\[
\begin{align*}
\min & \quad c_B^T x_B + c_N^T x_N \\
\text{s.t.} & \quad A_B x_B + A_N x_N = b \\
& \quad x_B, x_N \geq 0.
\end{align*}
\]

or

\[
\begin{align*}
\min & \quad c_B^T \left[ A_B^{-1} (b - A_N x_N) \right] + c_N^T x_N \\
\text{s.t.} & \quad x_B = A_B^{-1} (b - A_N x_N) \\
& \quad x_B, x_N \geq 0.
\end{align*}
\]
Rewriting the standard form

- With some more algebra, the LP becomes

\[
\begin{align*}
\min & \quad c^T A_B^{-1} b - (c^T A_B^{-1} A_N - c_N) x_N \\
\text{s.t.} & \quad x_B = A_B^{-1} b - A_B^{-1} A_N x_N \\
& \quad x_B, x_N \geq 0.
\end{align*}
\]

- By expressing the objective function by an equation with \( z \), the LP can be expressed as

\[
\begin{align*}
z & + (c^T A_B^{-1} A_N - c_N) x_N = c^T A_B^{-1} b \quad (0\text{th row}) \\
I x_B & + A_B^{-1} A_N x_N = A_B^{-1} b. \quad (1\text{st to } m\text{th row})
\end{align*}
\]
Rewriting the standard form

- What are we doing?
- Given a basis $B$, we update the system to make two things happen at the basic columns:
  - There is an identity matrix at the 1st to $m$th row:
    $$
    z + (c_B^T A_B^{-1} A_N - c_N^T) x_N = c_B^T A_B^{-1} b \quad \text{(0th row)}
    $$
    $$
    \boxed{I x_B} + A_B^{-1} A_N x_N = A_B^{-1} b. \quad \text{(1st to $m$th row)}
    $$
  - All numbers are zero at the 0th row:
    $$
    z + (c_B^T A_B^{-1} A_N - c_N^T) x_N = c_B^T A_B^{-1} b \quad \text{(0th row)}
    $$
    $$
    I x_B + A_B^{-1} A_N x_N = A_B^{-1} b. \quad \text{(1st to $m$th row)}
    $$
- Then we know what will happen when a nonbasic variable enters!
Improving the current bfs (the 2nd attempt)

- Recall that for the system

\[
\begin{align*}
  z + 2x_1 + 3x_2 &= 0 \\
  x_1 + 2x_2 + x_3 &= 6 \\
  2x_1 + x_2 + x_4 &= 8,
\end{align*}
\]

we start from \( x^1 = (0, 0, 6, 8) \) with \( z_1 = 0 \).
- For the basic columns (the 3rd and 4th ones), indeed we have the identity matrix and zeros.
- Then we know \( x_1 \) enters and \( x_4 \) leaves.
  - The basis becomes \( \{x_1, x_3\} \).
  - We need to update the system to

\[
\begin{align*}
  z + & \boxed{?x_2} + \boxed{x_3} + ?x_4 &= 0 \\
  + ?x_2 + & \boxed{x_3} + ?x_4 &= 6 \\
  + ?x_2 + & ?x_4 &= 8.
\end{align*}
\]

- How? **Elementary row operations!**
Updating the system

Starting from:

\[\begin{align*}
z &+ 2x_1 + 3x_2 &= 0 \quad (0) \\
x_1 &+ 2x_2 + x_3 &= 6 \quad (1) \\
2x_1 &+ x_2 + x_4 &= 8 \quad (2)
\end{align*}\]

- Multiply (2) by \(\frac{1}{2}\): \(x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_4 = 4\).
- Multiply (2) by \(-1\) and then add it into (1): \(\frac{3}{2}x_2 + x_3 - \frac{1}{2}x_4 = 2\).
- Multiply (2) by \(-1\) and then add it into (0): \(z + 2x_2 - x_4 = -8\).

Collectively, the system becomes

\[\begin{align*}
z &+ 2x_2 &- x_4 &= -8 \quad (0) \\
+ \frac{3}{2}x_2 &+ x_3 &- \frac{1}{2}x_4 &= 2 \quad (1) \\
x_1 &+ \frac{1}{2}x_2 &+ \frac{1}{2}x_4 &= 4 \quad (2)
\end{align*}\]
Improving the current bfs (finally!)

Given the updated system

\[
\begin{align*}
    z &+ 2x_2 - x_4 = -8 \quad (0) \\
    &+ \frac{3}{2}x_2 + x_3 - \frac{1}{2}x_4 = 2 \quad (1) \\
    x_1 &+ \frac{1}{2}x_2 + \frac{1}{2}x_4 = 4, \quad (2)
\end{align*}
\]

we now know how to do the next iteration.

- We are at \( x^2 = (4, 0, 2, 0) \) with \( z_2 = -8 \).
- One of \( x_2 \) and \( x_4 \) may enter.
- If \( x_2 \) enters, \( z \) will go down. Good!
- If \( x_4 \) enters, \( z \) will go up. Bad.

- Let \( x_2 \) enter:
  - Row 1: When \( x_2 \) goes up, \( x_3 \) goes down. \( x_2 \) can be as large as \( \frac{2}{3/2} = \frac{4}{3} \).
  - Row 2: When \( x_2 \) goes up, \( x_1 \) goes down. \( x_2 \) can be as large as \( \frac{4}{1/2} = 8 \).
  - So \( x_3 \) becomes 0 sooner than \( x_1 \). \( x_3 \) leaves the basis.
- The basic variables become \( x_1 \) and \( x_2 \). Let's update again.
Improving once more

Given the system

\[
\begin{align*}
z & \quad + \quad 2x_2 & \quad - \quad x_4 & \quad = \quad -8 \quad (0) \\
& \quad + \quad \frac{3}{2}x_2 & \quad + \quad x_3 & \quad - \quad \frac{1}{2}x_4 & \quad = \quad 2 \quad (1) \\
x_1 & \quad + \quad \frac{1}{2}x_2 & \quad + \quad \frac{1}{2}x_4 & \quad = \quad 4, \quad (2)
\end{align*}
\]

we now need to update it to fit the new basis \(\{x_1, x_2\}\).

- Multiply (1) by \(\frac{2}{3}\): \(x_2 + \frac{2}{3}x_3 - \frac{1}{3}x_4 = \frac{4}{3}\).
- Multiply (the updated) (1) by \(-\frac{1}{2}\) and add it to (2).
- Multiply (the updated) (1) by \(-2\) and add it to (0).

We get

\[
\begin{align*}
z & \quad - \quad \frac{4}{3}x_3 & \quad - \quad \frac{1}{3}x_4 & \quad = \quad -\frac{32}{3} \quad (0) \\
x_2 & \quad + \quad \frac{2}{3}x_3 & \quad - \quad \frac{1}{3}x_4 & \quad = \quad \frac{4}{3} \quad (1) \\
x_1 & \quad - \quad \frac{1}{3}x_3 & \quad + \quad \frac{2}{3}x_4 & \quad = \quad \frac{10}{3}. \quad (2)
\end{align*}
\]
No more improvement!

- The system

\[
\begin{align*}
    z & \quad - \quad \frac{4}{3} x_3 & \quad - \quad \frac{1}{3} x_4 & \quad = \quad -\frac{32}{3} & \quad (0) \\
    x_2 & \quad + \quad \frac{2}{3} x_3 & \quad - \quad \frac{1}{3} x_4 & \quad = \quad \frac{4}{3} & \quad (1) \\
    x_1 & \quad - \quad \frac{1}{3} x_3 & \quad + \quad \frac{2}{3} x_4 & \quad = \quad \frac{10}{3} & \quad (2)
\end{align*}
\]

tells us that the new bfs is \(x^3 = (\frac{10}{3}, \frac{4}{3}, 0, 0)\) with \(z_3 = -\frac{32}{3}\).

- Updating the system also gives us the new bfs and its objective value.

- Now... no more improvement is needed!

- Entering \(x_3\) makes things worse (\(z\) must go up).
- Entering \(x_4\) also makes things worse.

- \(x^3\) is an optimal solution.\(^9\) We are done!

---

\(^9\)This is indeed true, though a rigorous proof is omitted.
Visualizing the iterations

Let’s visualize this example and relate bfs with extreme points.

- The initial bfs corresponds to (0, 0).
- After one iteration, we move to (4, 0).
- After two iterations, we move to \((\frac{10}{3}, \frac{4}{3})\), which is optimal.

Please note that we move along edges to search among extreme points!
Summary

- To run the simplex method:
  - Find an initial bfs with its basis.\(^{10}\)
  - Among those nonbasic variables with positive coefficients in the 0th row, choose one to enter.\(^{11}\)
    - If there is none, terminate and report the current bfs as optimal.
  - According to the ratios from the basic and RHS columns, decide which basic variable should leave.\(^{12}\)
  - Find a new basis.
  - Make the system fit the requirements for basic columns:
    - Identity matrix in constraints (1st to \(m\)th row).
    - Zeros in the objective function (0th row).
  - Repeat.

---

\(^{10}\)How to find one?
\(^{11}\)What if there are multiple?
\(^{12}\)What if there is a tie? What if the denominator is 0 or negative?
The tableau representation

- Just as what we did for Gaussian eliminations, we typically omit variables when updating those systems.
- We organize coefficients into **tableaus**.
  - As the column with $z$ never changes, we do not include it in a tableau.
- For our example, the initial system

$$
\begin{align*}
  z + 2x_1 + 3x_2 &= 0 \\
  x_1 + 2x_2 + x_3 &= 6 \\
  2x_1 + x_2 + x_4 &= 8
\end{align*}
$$

can be expressed as

$$
\begin{array}{cccc|c}
2 & 3 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & x_3 = 6 \\
2 & 1 & 0 & 1 & x_4 = 8 \\
\end{array}
$$

- The basic columns have zeros in the 0th row and an identity matrix in the other rows.
- The identity matrix associates each row with a basic variable.
- A positive number in the 0th row of a nonbasic column means that variable can enter.
Using tableaus rather than systems

\[\begin{align*}
z &+ 2x_1 + 3x_2 &= 0 \\
x_1 + 2x_2 + x_3 &= 6 \\
2x_1 + x_2 + x_4 &= 8
\end{align*}\]

\[
\begin{bmatrix}
2 & 3 & 0 & 0 & | & 0 \\
1 & 2 & 1 & 0 & | & x_3 = 6 \\
2 & 1 & 0 & 1 & | & x_4 = 8
\end{bmatrix}
\]

\[
\begin{align*}
z &+ 2x_2 &- x_4 &= -8 \\
+ \frac{3}{2}x_2 + x_3 - \frac{1}{2}x_4 &= 2 \\
x_1 + \frac{1}{2}x_2 &+ \frac{1}{2}x_4 &= 4
\end{align*}\]

\[
\begin{bmatrix}
0 & 2 & 0 & -1 & | & -8 \\
0 & \frac{3}{2} & 1 & -\frac{1}{2} & | & x_3 = 2 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & | & x_1 = 4
\end{bmatrix}
\]

\[
\begin{align*}
z &- \frac{4}{3}x_3 - \frac{1}{3}x_4 &= -\frac{32}{3} \\
x_2 &+ \frac{2}{3}x_3 - \frac{1}{3}x_4 &= \frac{4}{3} \\
x_1 &- \frac{1}{3}x_3 + \frac{2}{3}x_4 &= \frac{10}{3}
\end{align*}\]

\[
\begin{bmatrix}
0 & 0 & -\frac{4}{3} & -\frac{1}{3} & | & -\frac{32}{3} \\
0 & 1 & \frac{2}{3} & -\frac{1}{3} & | & x_2 = \frac{4}{3} \\
1 & 0 & -\frac{1}{3} & \frac{2}{3} & | & x_1 = \frac{10}{3}
\end{bmatrix}
\]
The second example

Consider another example:

\[
\begin{align*}
\text{max} & \quad x_1 \\
\text{s.t.} & \quad 2x_1 - x_2 & \leq & \quad 4 \\
& \quad 2x_1 + x_2 & \leq & \quad 8 \\
& \quad x_2 & \leq & \quad 3 \\
& \quad x_i & \geq & \quad 0 \quad \forall \ i = 1, 2.
\end{align*}
\]

The standard form is

\[
\begin{align*}
\text{max} & \quad x_1 \\
\text{s.t.} & \quad 2x_1 - x_2 + x_3 = 4 \\
& \quad 2x_1 + x_2 + x_4 = 8 \\
& \quad x_2 + x_5 = 3 \\
& \quad x_i \geq 0 \quad \forall \ i = 1, \ldots, 5.
\end{align*}
\]
The first iteration

- We prepare the initial tableau. We have $x^1 = (0, 0, 4, 8, 3)$ and $z_1 = 0$.

\[
\begin{array}{ccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3 \\
\end{array}
\]

- For this \textbf{maximization} problem, we look for \textbf{negative} numbers in the 0th row. Therefore, $x_1$ enters.
  - Those numbers in the 0th row are called \textbf{reduced costs}.
  - The 0th row is $z - x_1 = 0$. Increasing $x_1$ can increase $z$.

- “Dividing the RHS column by the entering column” tells us that $x_3$ should leave (it has the minimum ratio).\footnote{The 0 in the 3rd row means that increasing $x_1$ does not affect $x_5$.}
  - This is called the \textbf{ratio test}. We \textbf{always} look for the smallest ratio.
The first iteration

- $x_1$ enters and $x_3$ leaves. The next tableau is found by pivoting at 2:

$$
\begin{array}{cccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
2 & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3 \\
\end{array} \rightarrow 
\begin{array}{cccccc|c}
0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 2 \\
1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\
0 & 2 & -1 & 1 & 0 & x_4 = 4 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3 \\
\end{array}
$$

- The new bfs is $x^2 = (2, 0, 0, 4, 3)$ with $z_2 = 2$.

- Continue?
  - There is a negative reduced cost in the 2nd column: $x_2$ enters.

- Ratio test:
  - That $-\frac{1}{2}$ in the 1st row shows that increasing $x_2$ makes $x_1$ larger. Row 1 does not participate in the ratio test.
  - For rows 2 and 3, row 2 wins (with a smaller ratio).
The second iteration

- $x_2$ enters and $x_4$ leaves. We pivot at 2.
- The second iteration is

\[
\begin{array}{cccc|c}
0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 2 \\
\hline
1 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\
0 & 2 & -1 & 1 & 0 & x_4 = 4 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3 \\
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 3 \\
\hline
1 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & x_1 = 3 \\
0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 & x_2 = 2 \\
0 & 0 & \frac{1}{2} & \frac{-1}{2} & 1 & x_5 = 1 \\
\end{array}
\]

- The third bfs is $x^3 = (3, 2, 0, 0, 1)$ with $z_3 = 3$.
  - It is optimal (why?).
  - Typically we write the optimal solution we find as $x^*$ and optimal objective value as $z^*$. 
Verifying our solution

- The three basic feasible solutions we obtain are
  - $x^1 = (0, 0, 4, 8, 3)$.
  - $x^2 = (2, 0, 0, 4, 3)$.
  - $x^3 = x^* = (3, 2, 0, 0, 1)$.

Do they fit our graphical approach?