Operations Research, Spring 2014

Inventory Theory

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Road map

- **Introduction**.
- The EOQ model.
- Variants of the EOQ model.
- The newsvendor model.

A news vendor's problem

TIME

- Time Inc. produces and sells over 150 magazines.
- At three different levels, one needs to decide **how many copies** to print/order:
  - Corporate level, wholesaler level, and retailer level.
- For each retailer, ordering **too many** or **too few** are both bad:
  - Too many: unsold copies are almost valueless.
  - Too few: potential sales are lost.
- **Demand randomness** is a big issue!
- For wholesalers and the corporate, the problems are harder:
  - The aggregate randomness is harder to estimate.
  - Bargaining and negotiation!
- Read the short story in Section 18.7 and the article on CEIBA.

What are inventory?

- For almost all firms producing or purchasing products to sell, they need **inventory**.
  - If each batch of production or procurement requires some **fixed costs**, increasing the batch size saves money.
  - If **demand is uncertain**, we want a buffer for supply-demand mismatch.
- Key questions in the manufacturing and retailing industries regarding inventory include:
  - When to do replenishment?
  - How much to replenish?
  - From which suppliers?
- We will introduce basic OR models for optimizing inventory decisions.
  - They are direct applications of NLP.
- Read Sections 18.1–18.3 and 18.7 in the textbook.

Categories of inventory models

- There are two kinds of inventory systems:
  - In a **periodic review** system, orders are placed (productions are initiated) once per "period".
  - In a **continuous review** system, one may replenish at any time point.
- The demands may be either **deterministic** or **random** (stochastic).
- There are four categories of inventory problems:

Demand	Review time	
	Periodic	Continuous
Deterministic	1	2
Random	3	4

An LP-based inventory model

- We have seen a periodic review system for deterministic demands:
  - We have T periods with different demands.
  - In each period, we first produce and then sell.
  - Unsold products become ending inventories.
  - We want to minimize the total cost.
  - In period t, C_t is the unit production cost, D_t is the unit production quantity, and H is the unit holding cost per period.
- The formulation is

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{T} (C_t x_t + H y_t) \\
\text{s.t.} \quad & y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., T \\
& y_0 = 0 \\
& x_t, y_t \geq 0 \quad \forall t = 1, ..., T.
\end{aligned}
$$

Two NLP-based inventory models

- We will introduce two NLP-based inventory models:
  - The **economic order quantity** (EOQ) model.
  - The **newsvendor** model.
- They are the foundations of most advanced inventory models.
- Each of them fits one category:

Demand	Review time	
	Periodic	Continuous
Deterministic	The LP-based model	EOQ
Random	Newsvendor	(Beyond the scope)

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Motivating example

- IM Airline uses 500 taillights per year. It purchases these taillights from a manufacturer at a unit price \$500.
- Taillights are consumed at a **constant rate** throughout a year.
- Whenever IM Airline places an order, an **ordering cost** of \$5 is incurred regardless of the order quantity.
- The **holding cost** is 2 cents per taillight per month.
- IM Airline wants to minimize the total cost, which is the sum of ordering, purchasing, and holding costs.
- How much to order? When to order?
  - What is the benefit of having a small or large order?

The EOQ model

- IM Airline's question may be answered with the economic order quantity (EOQ) model.
- We look for the order quantity that is the most economic.
  - We look for a **balance** between the ordering cost and holding cost.
- Technically, we will formulate an NLP whose optimal solution is the optimal order quantity.
- Assumptions for the (most basic) EOQ model:
  - Demand is deterministic and occurs at a constant rate.
  - Regardless the order quantity, a fixed ordering cost is incurred.
  - No shortage is allowed.
  - The ordering lead time is zero.
  - The inventory holding cost is constant.

Parameters and the decision variable

- Parameters:

$$\begin{aligned} D &= \text{annual demand (units)}, \\ K &= \text{unit ordering cost (\$)}, \\ h &= \text{unit holding cost per year (\$), and} \\ p &= \text{unit purchasing cost (\$)}. \end{aligned}$$

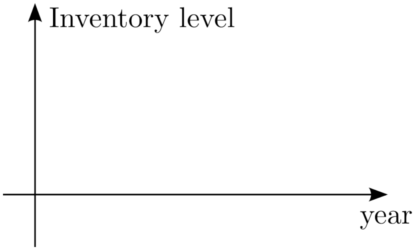
- Decision variable:

$$q = \text{order quantity per order (units)}.$$

- Objective: Minimizing annual total cost.
- For all our calculations, we will use **one year** as our time unit. Therefore, D can be treated as the demand **rate**.

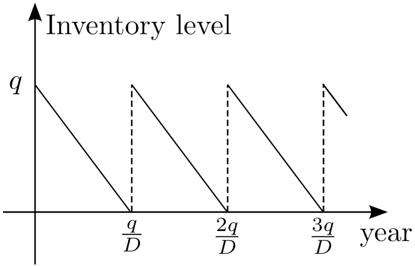
Inventory level

- To formulate the problem, we need to understand how the **inventory level** is affected by our decision.
  - The number of inventory we have on hand.
- Because there is no ordering lead time, we will always place an order when the inventory level is zero.
- As inventory is consumed at a constant rate, the inventory level will change by time like this:



Inventory level by time

- The same situation will **repeat** again and again:



- In average, how many units are stored?

Annual costs

- Annual holding cost $= h \times \frac{q}{2} = \frac{hq}{2}$.
  - For one year, the length of the time period is 1 and the inventory level is $\frac{q}{2}$ **in average**.
- Annual purchasing cost $= pD$.
  - We need to buy D units regardless the order quantity q.
- Annual ordering cost $= K \times \frac{D}{q} = \frac{KD}{q}$.
  - The number of orders in a year is $\frac{D}{q}$.
- The NLP for optimizing the ordering decision is

$$\min_{q \geq 0} \ \frac{KD}{q} + pD + \frac{hq}{2}.$$

- As pD is just a constant, we will ignore it and let $TC(q) = \frac{KD}{q} + \frac{hq}{2}$ be our objective function.

Convexity of the EOQ model

- For

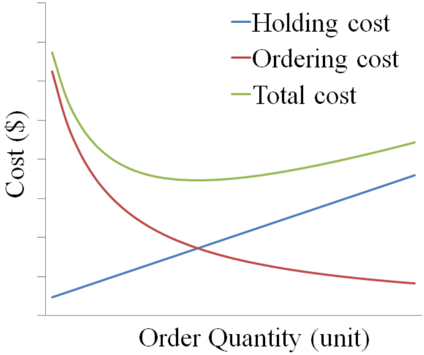
$$TC(q) = \frac{KD}{q} + \frac{hq}{2},$$

we have

$$TC'(q) = -\frac{KD}{q^2} + \frac{h}{2} \text{ and}$$

$$TC''(q) = \frac{2KD}{q^3} > 0.$$

Therefore, $TC(q)$ is convex in q.



Optimizing the order quantity

- Let q^* be the quantity satisfying the FOC:

$$TC'(q^*) = -\frac{KD}{(q^*)^2} + \frac{h}{2} = 0 \quad \Rightarrow \quad q^* = \sqrt{\frac{2KD}{h}}.$$

- As this quantity is feasible, it is optimal.
- The resulting annual holding and ordering cost is $TC(q^*) = \sqrt{2KDh}$.
- The optimal order quantity q^* is called the **EOQ**. It is:
  - Increasing in the ordering cost K.
  - Increasing in the annual demand D.
  - Decreasing in the holding cost h.

  Why?

Example

- IM Airline uses 500 taillights per year.
- The ordering cost is $5 per order.
- The holding cost is 2 cents per unit per month.
- Taillights are consumed at a constant rate.
- No shortage is allowed.
- Questions:
  - What is the EOQ?
  - How many orders to place in each year?
  - What is the order cycle time (time between two orders)?

Example: the optimal solution

- The EOQ is

$$q^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2(5)(500)}{(0.24)}} \approx \sqrt{20833.33} \approx 144.34 \text{ units.}$$

  - Make sure that time units are consistent!
  - 2 cents per unit per month = \$0.24 per unit per year.
- The average number of orders in a year is $\frac{500}{q^*} \approx 3.464$ orders.
- The order cycle time is

$$T^* = \frac{1}{3.464} \approx 0.289 \text{ year} \approx 3.464 \text{ months.}$$

- The number of orders in a year and the order cycle time are the same! Is it a coincidence?

Example: cost analysis

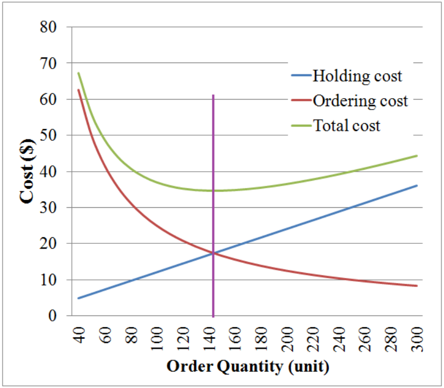
- The EOQ is $q^* \approx 144.34$ units.
- The annual holding cost is

$$\frac{hq^*}{2} \approx \$17.32.$$

- The annual ordering cost is

$$\frac{KD}{q^*} \approx \$17.32.$$

- The two costs are identical! Is it a coincidence?



Road map

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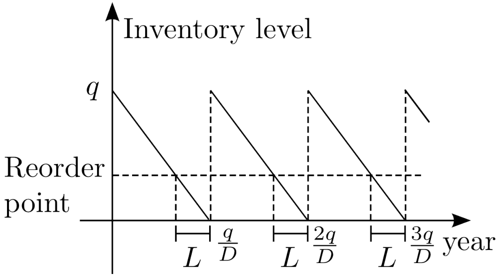
- **Variants of the EOQ model**.
  - Ordering lead time.
  - Economic production quantity.
- The newsvendor model.

Nonzero lead time

- What if there is an **ordering lead time** $L > 0$?
  - After we place an order, we will receive the product after L year.
- In this case, we want to calculate the **reorder point**: the inventory level at which an order should be placed.



Reorder points

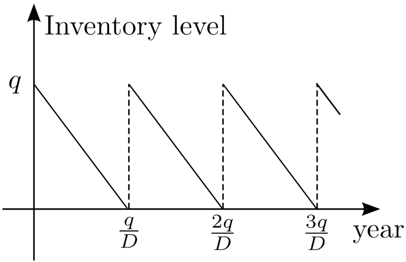
- When to order?
- Let R be the reorder point. We want to calculate R such that we receive products exactly when we have **no inventory**.
- If $L \leq T^*$:

$$R = LD.$$

  - T^* is the order cycle time.
  - L must be measured in years!
- If $L \geq T^*$:

$$R = D(L - kT^*)$$

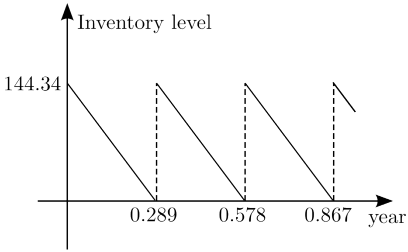
  for some $k \in \mathbb{N}$ such that $0 \leq L - kT^* \leq T^*$.



Example

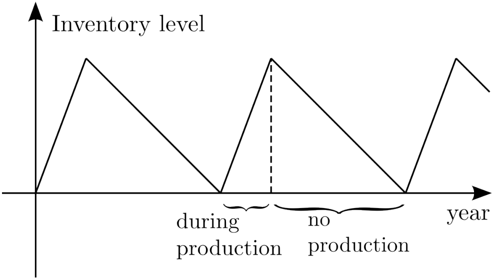
- For IM Airline, suppose the **ordering lead time** is 1 month:
  - The EOQ is $q^* \approx 144.34$ units. The optimal cycle time $T^* = 0.289$ years.
  - The demand rate $D = 500$ units. The lead time is $L = \frac{1}{12} \approx 0.083$ year.
- What is the reorder point?
  - Because $L < T^*$, we have $R = LD = \frac{500}{12} \approx 41.67$ units.
- What if the lead time is 4 months?
  - Lead time: $L = \frac{4}{12} \approx 0.333$ years.
  - Because $L > T^*$ and $L - T^* < T^*$, we have

$$R = D(L - T^*)$$
$$\approx 500 \times (0.333 - 0.289)$$
$$= 500 \times 0.044 = 22 \text{ units.}$$



Economic production quantity (EPQ)

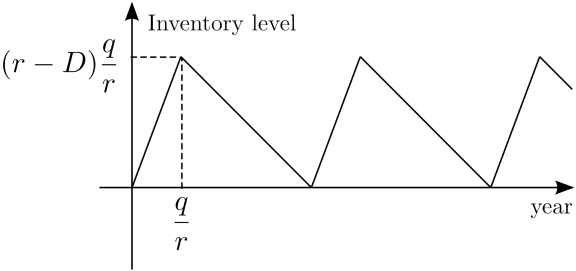
- When products are produced rather than purchased, typical they are "received" at a continuous rate.
- The inventory level now looks like:



- The model that finds the optimal **production lot size** is called the **economic production quantity** (EPQ) model.
- Under the assumption that the product is **produced at a constant rate** of r units per year, what lot size minimizes the total cost?

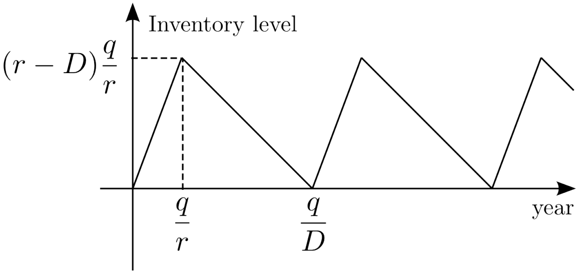
Economic production quantity

- Suppose we choose q as our production lot size.
- During the **up time**, the inventory level increases at the rate $r - D$.
  - While we produce at the rate r, we also consume at the rate D.
- The length of the up time is $\frac{q}{r}$ year. Why?
- So the maximum inventory level (achieved at the end of a up period) is $(r - D)\frac{q}{r}$.



Economic production quantity

- Still, the amount we produce in a lot will be depleted in $\frac{q}{D}$ year.
  - The period with no production is called the **down time**.



- Key question: What is the **average inventory level**?

Economic production quantity

- The annual holding cost now becomes $h\left[\frac{q(r-D)}{2r}\right]$.
- The annual setup cost is still $K(\frac{D}{q})$.
- The total annual holding and setup cost is:

$$\frac{hq(r-D)}{2r} + \frac{KD}{q}.$$

- Note that this is the same as the EOQ model $(\frac{hq}{2} + \frac{KD}{q})$ if we let $h(\frac{r-D}{r}) = h(1 - \frac{D}{r})$ be the **effective holding cost**.
- The optimal production lot size (the EPQ) is thus

$$q^* = \sqrt{\frac{2KD}{h\left(1 - \frac{D}{r}\right)}}.$$

Example

- IM Auto needs to produce 10000 cars per year.
- Each car requires \$2000 to produce.
- Each run requires \$200 to set up.
- The production rate is 25000 cars per year.
- Annual **holding cost rate** is 25%:
  - The holding cost per car per year is $\frac{\$2000}{4} = \$500$.
- What is the EPQ and optimal cycle time?

Example

- The EPQ is

$$\sqrt{\frac{2KD}{h(1-\frac{D}{r})}} = \sqrt{\frac{2(200)(10000)}{500(1-\frac{10000}{25000})}} = 115.47 \text{ cars.}$$

- The optimal cycle time is

$$\frac{1}{\frac{D}{T^*}} = \frac{1}{\frac{10000}{115.47}} \approx 0.012 \text{ year } \approx 4.21 \text{ days.}$$

- Some questions:
  - Will the annual holding cost and setup cost still be identical? Why?
  - What if there is a setup time?
  - In each year, how much time is the up time?

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Newsvendor model

- In some situations, people sell **perishable products**.
  - They become valueless after the **selling season** is end.
  - E.g., newspapers become valueless after each day.
  - High-tech goods become valueless once the next generation is offered.
  - Fashion goods become valueless when they become out of fashion.
- In many cases, the seller only have **one chance** for replenishment.
  - E.g., a small newspaper seller can order only once and obtain those newspapers only at the morning of each day.
- Often sellers of perishable products face **uncertain demands**.
- **How many** products one should prepare for the selling season?
  - Not too many and not too few!

Newsvendor model

- Let D be the uncertain demand (so D is a **random variable**).
- Let F and f be the cdf and pdf of D (assuming D is continuous).
  - If D is uniformly distributed between 0 and 100, we have $f(x) = \frac{1}{100}$ and $F(x) = \Pr(D \leq x) = \frac{x}{100}$ for all $x \in [0, 100]$.
  - If D is normally distributed with mean 50 and standard deviation 10:

Overage and underage costs

- Let c_o be the **overage cost** and c_u be the **underage cost**.
  - They are also called overstocking and understocking costs.
  - They are the costs for preparing too many or too few products.
- Components of overage and underage costs may include:
  - Sales revenue r for each unit sold.
  - Purchasing cost c for each unit purchased.
  - Salvage value v for each unit unsold.
  - Disposal fee d for each unit unsold.
  - Shortage cost (loss of goodwill) s for each unit of shortage.
- With these quantities, we have
  - The overage cost $c_o = c + d - v$.
  - The underage cost $c_u = r - c + s$.
- What is an optimal order quantity?
  - As demands are uncertain, we try to minimize the **expected** total overage and underage costs.

Formulation of the newsvendor problem

- Let q be the order quantity (inventory level).
- Let x be the **realization** of demand.
  - D is a random variable and x is a realized value of D.
- Then the realized overage or underage cost is

$$c(q,x) = \begin{cases} c_o(q-x) & \text{if } q \geq x \\ c_u(x-q) & \text{if } q < x \end{cases}$$

  or simply $c(q,x) = c_o(q-x)^+ + c_u(x-q)^+$, where $y^+ = \max(y,0)$.
- Therefore, the **expected total cost** is

$$c(q,D) = \mathbb{E}\Big[c_o(q-D)^+ + c_u(D-q)^+\Big].$$

- We want to find a quantity q that solves the NLP

$$\min_{q \geq 0} \ \mathbb{E}\Big[c_o(q-d)^+ + c_u(d-q)^+\Big].$$

Convexity of the cost function

- The cost function $c(q, D) = \mathbb{E}\Big[c_o(q-D)^+ + c_u(D-q)^+\Big]$.
- By assuming that D is continuous, the cost function $c(q, D)$ is

$$
\begin{aligned}
&\int_0^\infty \Big[c_o(q-x)^+ + c_u(x-q)^+\Big] f(x)dx \\
&= \int_0^q \Big[c_o(q-x) + c_u \cdot 0\Big] f(x)dx + \int_q^\infty \Big[c_o \cdot 0 + c_u(x-q)\Big] f(x)dx \\
&= c_o \int_0^q (q-x)f(x)dx + c_u \int_q^\infty (x-q)f(x)dx \\
&= c_o\left[q\int_0^q f(x)dx - \int_0^q xf(x)dx\right] + c_u\left[\int_q^\infty xf(x)dx - q\int_q^\infty f(x)dx\right] \\
&= c_o\left[qF(q) - \int_0^q xf(x)dx\right] + c_u\left[\int_q^\infty xf(x)dx - q\big(1-F(q)\big)\right].
\end{aligned}
$$

Convexity of the cost function

- We have

$$c(q,D) = c_o\left[qF(q) - \int_0^q xf(x)dx\right] + c_u\left[\int_q^\infty xf(x)dx - q\big(1-F(q)\big)\right].$$

- The first-order derivative of $c(q,D)$ is

$$\begin{aligned} & c'(q,D) \\ = {} & c_o\Big[F(q) + qf(q) - qf(q)\Big] + c_u\Big[-qf(q) - (1-F(q)) + qf(q)\Big] \\ = {} & c_o\big[F(q)\big] - c_u\big[1-F(q)\big]. \end{aligned}$$

- The second-order derivative of $c(q,D)$ is

$$c''(q,D) = c_o f(q) + c_u f(q) = f(q)(c_u + c_o) > 0.$$

- So $c(q,D)$ is convex in q.

Optimizing the order quantity

- Let q^* be the order quantity that satisfies the FOC, we have

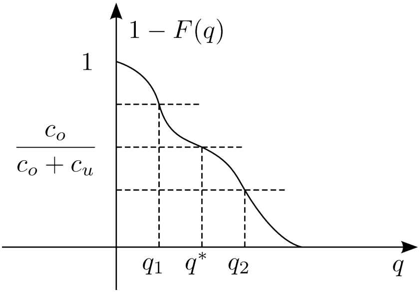
$$c_o F(q^*) - c_u(1 - F(q^*)) = 0$$
$$\Rightarrow F(q^*) = \frac{c_u}{c_o + c_u} \quad \text{or} \quad 1 - F(q^*) = \frac{c_o}{c_o + c_u}.$$

- Such q^* must be positive (for regular demand distributions).
  - So q^* is optimal.
  - The quantity q^* is called the **newsvendor** quantity.
  - Note that the only assumption we made is that D is continuous!
- Note that to minimize the expected total cost, the seller should **intentionally** create some shortage!
  - The optimal probability of having a shortage is $1 - F(q^*) = \frac{c_o}{c_o + c_u}$.

Interpretations of the newsvendor quantity

- The probability of having a shortage, $1 - F(q)$, is decreasing in q.
- The newsvendor quantity q^* satisfies $1 - F(q^*) = \frac{c_o}{c_o + c_u}$.
- The optimal quantity q^* is:
  - Decreasing in c_o.
  - Increasing in c_u.

  Why?



Example 1

- Suppose for a newspaper:
  - The unit purchasing cost is \$5.
  - The unit retail price is \$15.
  - The demand is uniformly distributed between 20 to 50.
- Overage cost $c_o = 5$ and underage cost $c_u = 15 - 5 = 10$.
- The optimal order quantity q^* satisfies

$$1 - F(q^*) = \left(1 - \frac{q^* - 20}{50 - 20}\right) = \frac{5}{5 + 10} \quad \Rightarrow \quad \frac{50 - q^*}{30} = \frac{1}{3},$$

  which implies $q^* = 40$.
- If the unit purchasing cost decreases to \$4, we need $\frac{50-q^{**}}{30} = \frac{4}{15}$ and thus $q^{**} = 42$.
  - As the purchasing cost decreases, we **prefer overstocking** more. Therefore, we stock more.

Example 2

- Suppose for one kind of apple:
  - The unit purchasing cost is \$15, the unit retail price is \$21, and the unit salvage value is \$1.
  - The demand is normally distributed with mean 90 and standard deviation 20.
  - Overage cost $c_o = 15 - 1 = 14$ and underage cost $c_u = 21 - 15 = 6$.
- The optimal order quantity q^* satisfies

$$\Pr(D < q^*) = \frac{6}{14+6} \quad \Rightarrow \quad \Pr\left(Z < \frac{q^* - 90}{20}\right) = 0.3,$$

  where $Z \sim \text{ND}(0,1)$.
- By looking at a probability table or using a software, we find $\Pr(Z < -0.5244) = 0.3$. Therefore, $\frac{q^* - 90}{20} = -0.5244$ and $q^* = 79.512$.
  - As the purchasing cost is so high, we want to **reject more than half** of the consumers!