Operations Research, Spring 2016
Suggested Solution for Homework 1

Solution providers: Kiwi Liu and Johnny Chen
Department of Information Management
National Taiwan University

1. (a) We number city 1-4 Taipei, Kaohsiung, Taoyuan, Hsinchu and assume month 0 be the beginning of month 1.

   Let the parameters be

   D_{ij} = the demands for air conditioners of month j at city i, $i = 1, ..., 4$, $j = 1, ..., 6$,
   C_{ij} = the shipping cost from city j to city i, $i = 1, 2$, $j = 3, 4$.

   Let the decision variables be

   p_{ij} = production quantity of month j in city i, $i = 3, 4$, $j = 1, ..., 6$,
   h_{ij} = the amount of air coditioners of month j shipping from Hsinchu to city i, $i = 1, 2$, $j = 1, ..., 6$,
   t_{ij} = the amount of air coditioners of month j shipping from Taoyuan to city i, $i = 1, 2$, $j = 1, ..., 6$,
   x_{ij} = ending inventory of month j at city i, $i = 3, 4$, $j = 0, ..., 6$.

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{6}(400p_{3j} + 350p_{4j} + \sum_{i=1}^{2} C_{i3}h_{ij} + \sum_{i=1}^{2} C_{i4}t_{ij} + 80\sum_{i=3}^{4} x_{ij}) \\
\text{s.t.} \quad & x_{30} = 1000 \\
& x_{3j-1} + p_{3j} - \sum_{i=1}^{2} h_{ij} = x_{3j} \quad \forall j = 1, ..., 6 \\
& x_{40} = 0 \\
& x_{4j-1} + p_{4j} - \sum_{i=1}^{2} t_{ij} = x_{4j} \quad \forall j = 1, ..., 6 \\
& h_{ij} + t_{ij} = D_{ij} \quad \forall i = 1, 2, \ j = 1, ...6 \\
& 2p_{3j} \leq 4000 \quad \forall j = 1, ..., 6 \\
& 2.5p_{4j} \leq 4000 \quad \forall j = 1, ..., 6 \\
& p_{ij} \geq 0 \quad \forall i = 3, 4, \ j = 1, ..., 6 \\
& h_{ij} \geq 0, \quad t_{ij} \geq 0 \quad \forall i = 1, 2, \ j = 1, ..., 6 \\
& x_{ij} \geq 0 \quad \forall i = 3, 4, \ j = 0, ..., 6.
\end{aligned}
$$

   (b) The setting is the same as 1.(a)
   Let the parameters be

   D_{ij} = the demands for air conditioners of month j at city i, $i = 1, ..., 4$, $j = 1, ..., 6$,
   C_{ij} = the shipping cost from city j to city i, $i = 1, 2$, $j = 3, 4$.

   Let the decision variables be

   s_{ij} = sales quantity of month j in city i, $i = 3, 4$, $j = 1, ...6$,
   p_{ij} = production quantity of month j in city i, $i = 3, 4$, $j = 1, ..., 6$,
   h_{ij} = the amount of air coditioners of month j shipping from Hsinchu to city i, $i = 1, 2$, $j = 1, ..., 6$,
   t_{ij} = the amount of air coditioners of month j shipping from Taoyuan to city i, $i = 1, 2$, $j = 1, ..., 6$,
   x_{ij} = ending inventory of month j at city i, $i = 3, 4$, $j = 0, ..., 6$.

$$\begin{aligned}
\min \quad & \sum_{j=1}^{6}(600\sum_{i=3}^{4} s_{ij} - 400p_{3j} - 350p_{4j} - \sum_{i=1}^{2} C_{i3}h_{ij} - \sum_{i=1}^{2} C_{i4}t_{ij} - 80\sum_{i=3}^{4} x_{ij}) \\
\text{s.t.} \quad & x_{30} = 1000 \\
& x_{3j-1} + p_{3j} - \sum_{i=1}^{2} h_{ij} = x_{3j} \quad \forall j = 1, ..., 6 \\
& x_{40} = 0 \\
& x_{4j-1} + p_{4j} - \sum_{i=1}^{2} t_{ij} = x_{4j} \quad \forall j = 1, ..., 6 \\
& h_{ij} + t_{ij} = s_{ij} \quad \forall i = 1, 2, \ j = 1, ...6 \\
& s_{ij} \leq D_{ij} \quad \forall i = 1, 2, \ j = 1, ...6 \\
& 2p_{3j} \leq 4000 \quad \forall j = 1, ..., 6 \\
& 2.5p_{4j} \leq 4000 \quad \forall j = 1, ..., 6 \\
& p_{ij} \geq 0 \quad \forall i = 3, 4, \ j = 1, ..., 6 \\
& s_{ij} \geq 0, \quad h_{ij} \geq 0, \quad t_{ij} \geq 0 \quad \forall i = 1, 2, \ j = 1, ..., 6 \\
& x_{ij} \geq 0 \quad \forall i = 3, 4, \ j = 0, ..., 6.
\end{aligned}$$

2. (a) The feasible region and isoquant line are illustrated in Figure 1. It is clear that we should push the isoquant line until we stop at the extreme point $(6, 2)$, which is an optimal solution.

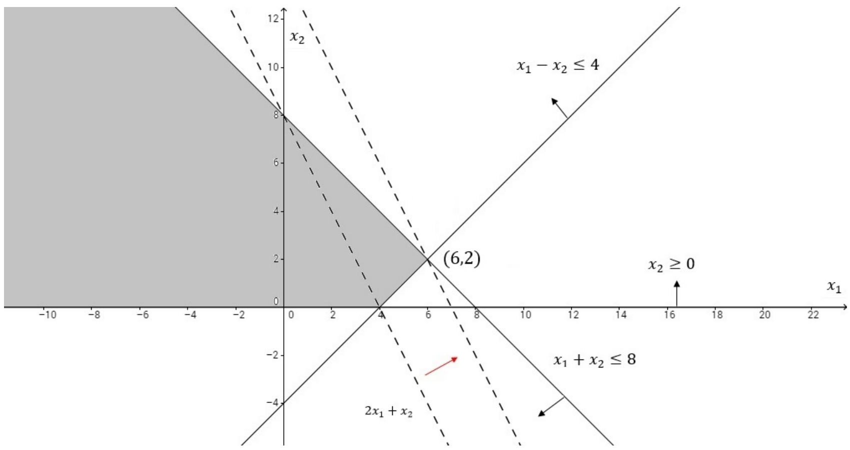

Figure 1: Graphical solution for Problem 2a

(b) The standard form is

$$\begin{aligned}
\max \quad & 2x_1 - 2x_3 + x_2 \\
\text{s.t.} \quad & x_1 - x_3 - x_2 + x_4 = 4 \\
& x_1 - x_3 + x_2 + x_5 = 8 \\
& x_i \geq 0 \quad \forall i = 1, ..., 5.
\end{aligned}$$

Since in the standard form we have five variables and two constraints, there should be two basic variables and three nonbasic variables in a basic solution. The ten possible ways to choose three (nonbasic) variables to be 0 are listed in the table below.

x_1	x_2	x_3	x_4	x_5	basis	Basic feasible solution?
6	2	0	0	0	$\{x_1, x_2\}$	Yes
–	0	–	0	0	$\{x_1, x_3\}$	No
8	0	0	-4	0	$\{x_1, x_4\}$	No
4	0	0	0	4	$\{x_1, x_5\}$	Yes
0	2	-6	0	0	$\{x_2, x_3\}$	No
0	8	0	12	0	$\{x_2, x_4\}$	Yes
0	-4	0	0	12	$\{x_2, x_5\}$	No
0	0	-8	-4	0	$\{x_3, x_4\}$	No
0	0	-4	0	4	$\{x_3, x_5\}$	No
0	0	0	4	8	$\{x_4, x_5\}$	Yes

(c) The initial tableau is

$$\begin{array}{ccccc|c} -2 & -1 & 2 & 0 & 0 & 0 \\ \hline 1 & -1 & -1 & 1 & 0 & x_4 = 4 \\ 1 & 1 & -1 & 0 & 1 & x_5 = 8 \end{array}$$

We use smallest index rule and run four iterations to get

$$\begin{array}{ccccc|c} -2 & -1 & 2 & 0 & 0 & 0 \\ \hline \boxed{1} & -1 & -1 & 1 & 0 & x_4 = 4 \\ 1 & 1 & -1 & 0 & 1 & x_5 = 8 \end{array} \rightarrow \begin{array}{ccccc|c} 0 & -3 & 0 & 2 & 0 & 8 \\ \hline 1 & -1 & -1 & 1 & 0 & x_1 = 4 \\ 0 & \boxed{2} & 0 & -1 & 1 & x_5 = 4 \end{array}$$

$$\rightarrow \begin{array}{ccccc|c} 0 & 0 & 0 & \frac{1}{2} & \frac{3}{2} & 14 \\ \hline 1 & 0 & -1 & \frac{1}{2} & \frac{1}{2} & x_1 = 6 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & x_2 = 2 \end{array}$$

an optimal solution to the original LP is $(x_1^*, x_2^*) = (6, 2)$ with objective value $z^* = 14$.

(d) The original LP becomes

$$\begin{aligned} \max \quad & x_1 - x_3 + 2x_2 \\ \text{s.t.} \quad & x_1 - x_3 - x_2 + x_4 = 4 \\ & 2x_1 - x_3 + x_2 + x_5 = 8 \\ & x_i \geq 0 \quad \forall i = 1, ..., 5. \end{aligned}$$

The initial tableau is

$$\begin{array}{ccccc|c} -1 & -2 & 1 & 0 & 0 & 0 \\ \hline 1 & -1 & -1 & 1 & 0 & x_4 = 4 \\ 1 & 1 & -1 & 0 & 1 & x_5 = 8 \end{array}$$

We use smallest index rule and run four iterations.

-1	-2	1	0	0	0
$\boxed{1}$	-1	-1	1	0	$x_4 = 4$
1	1	-1	0	1	$x_5 = 8$

$\rightarrow$

0	-3	0	1	0	4
1	-1	-1	1	0	$x_1 = 4$
0	$\boxed{2}$	0	-1	1	$x_5 = 4$

$\rightarrow$

0	0	0	$-\frac{1}{2}$	$\frac{3}{2}$	10
1	0	-1	$\boxed{\frac{1}{2}}$	$\frac{1}{2}$	$x_1 = 6$
0	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	$x_2 = 2$

$\rightarrow$

1	0	-1	0	2	16
2	0	-2	1	1	$x_1 = 12$
1	1	-1	0	1	$x_2 = 8$

We can notice that x_3 is the only variable with negative coefficient in 1st row while its coefficient are all negative in other rows. It means that the constraint is unbounded, so we can't find the objective value in this modified LP.

3. (a) The standard form LP is

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_4 = 4 \\ & x_1 + 2x_2 + 3x_3 + x_5 = 9 \\ & x_3 - x_6 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 6. \end{aligned}$$

We need to use two-phase implementation.

i. The Phase-I standard form LP is

$$\begin{aligned} \min \quad & x_7 \\ \text{s.t.} \quad & x_1 + x_2 + x_4 = 4 \\ & x_1 + 2x_2 + 3x_3 + x_5 = 9 \\ & x_3 - x_6 + x_7 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 7. \end{aligned}$$

First, solve the Phase-I LP with smallest index rule which tries to minimize x_7.

0	0	0	0	0	0	-1	0
1	1	0	1	0	0	0	$x_4 = 4$
1	2	3	0	1	0	0	$x_5 = 9$
0	0	1	0	0	-1	1	$x_7 = 3$

$\overset{adjust}{\overbrace{\rightarrow}}$

0	0	1	0	0	-1	0	3
1	1	0	1	0	0	0	$x_4 = 4$
1	2	$\boxed{3}$	0	1	0	0	$x_5 = 9$
0	0	1	0	0	-1	1	$x_7 = 3$

$\rightarrow$

$-\frac{1}{3}$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	-1	0	0
1	1	0	1	0	0	0	$x_4 = 4$
$\frac{1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$x_3 = 3$
$-\frac{1}{3}$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	-1	1	$x_7 = 0$

ii. According to the phase I, we get a feasible solution $x' = (0, 0, 3, 4, 0, 0, 0)$. We can know that there are multiple solutions. Before doing phase II, we should let x_7 leave basis. Here, we choose x_6 to be entering variable and make an adjustment.

$-\frac{1}{3}$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	-1	0	0
1	1	0	1	0	0	0	$x_4 = 4$
$\frac{1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$x_3 = 3$
$-\frac{1}{3}$	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	-1	1	$x_7 = 0$

$\overset{adjust}{\overbrace{\rightarrow}}$

0	0	0	0	0	0	0
1	1	0	1	0	0	$x_4 = 4$
$\frac{1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	$x_3 = 3$
$\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	1	$x_6 = 0$

Then, solve the Phase-II LP.We use smallest index rule and run four iterations to get

-3	-2	-1	0	0	0	0
1	1	0	1	0	0	$x_4 = 4$
$\frac{1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	$x_3 = 3$
$\frac{1}{3}$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	1	$x_6 = 0$

$\overset{adjust}{\overbrace{\rightarrow}}$

$-\frac{8}{3}$	$-\frac{4}{3}$	0	0	$\frac{1}{3}$	0	3
1	1	0	1	0	0	$x_4 = 4$
$\frac{1}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	$x_3 = 3$
$\boxed{\frac{1}{3}}$	$\frac{2}{3}$	0	0	$\frac{1}{3}$	1	$x_6 = 0$

$\rightarrow$

0	4	0	0	3	8	3
0	-1	0	1	-1	-3	$x_4 = 4$
0	0	1	0	0	-1	$x_3 = 3$
1	2	0	0	1	3	$x_1 = 0$

an optimal solution to the LP is $(x_1^*, x_2^*, x_3^*) = (0, 0, 3)$ with objective value $z^* = 3$. There isn't any iteration that has no improvement.

4. (a) The feasible region and isoquant line are illustrated in Figure 2. It is clear that we should push the isoquant line until we stop at the extreme point $(4, 6)$, which is an optimal solution. The optimal value is 22.

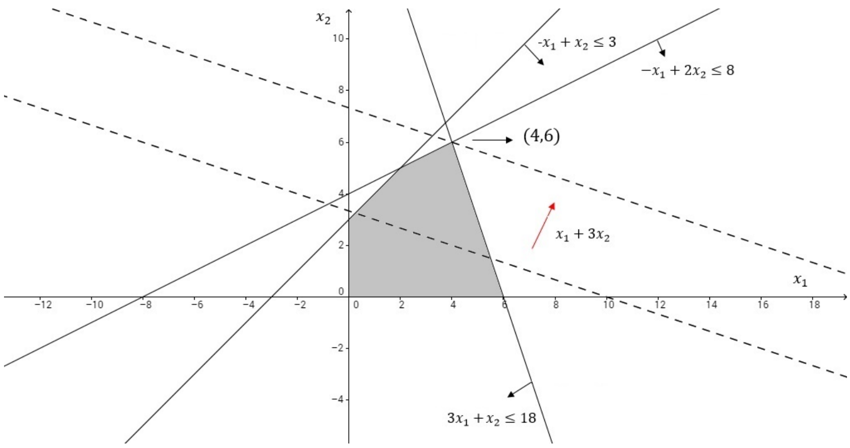

Figure 2: Graphical solution for Problem 4a

(b) The standard form is

$$\begin{aligned} \max \quad & x_1 + 3x_2 \\ \text{s.t.} \quad & -x_1 + x_2 + x_3 = 3 \\ & -x_1 + 2x_2 + x_4 = 8 \\ & 3x_1 + x_2 + x_5 = 18 \\ & x_i \geq 0 \quad \forall i = 1, ..., 5. \end{aligned}$$

The initial tableau is

-1	-3	0	0	0	0
-1	1	1	0	0	$x_3 = 3$
-1	2	0	1	0	$x_4 = 8$
3	1	0	0	1	$x_5 = 18$

We use "highest-magnitude reduced cost rule" and run four iterations.

$$
\begin{array}{ccccc|c}
-1 & -3 & 0 & 0 & 0 & 0 \\ \hline
-1 & \boxed{1} & 1 & 0 & 0 & x_3 = 3 \\
-1 & 2 & 0 & 1 & 0 & x_4 = 8 \\
3 & 1 & 0 & 0 & 1 & x_5 = 18
\end{array}
\rightarrow
\begin{array}{ccccc|c}
-4 & 0 & 3 & 0 & 0 & 9 \\ \hline
-1 & 1 & 1 & 0 & 0 & x_2 = 3 \\
\boxed{1} & 0 & -2 & 1 & 0 & x_4 = 2 \\
4 & 0 & -1 & 0 & 1 & x_5 = 15
\end{array}
$$

$$
\rightarrow
\begin{array}{ccccc|c}
0 & 0 & -5 & 4 & 0 & 17 \\ \hline
0 & 1 & -1 & 1 & 0 & x_2 = 5 \\
1 & 0 & -2 & 1 & 0 & x_1 = 2 \\
0 & 0 & \boxed{7} & -4 & 1 & x_5 = 7
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & 0 & 0 & \frac{8}{7} & \frac{5}{7} & 22 \\ \hline
0 & 1 & 0 & \frac{3}{7} & \frac{1}{7} & x_2 = 6 \\
1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & x_1 = 4 \\
0 & 0 & 1 & -\frac{4}{7} & \frac{1}{7} & x_3 = 1
\end{array}
$$

an optimal solution to the LP is $(x_1^*, x_2^*) = (4, 6)$ with objective value $z^* = 22$. There isn't any iteration that has no improvement.

(c) The initial tableau is

$$
\begin{array}{ccccc|c}
-1 & -3 & 0 & 0 & 0 & 0 \\ \hline
-1 & 1 & 1 & 0 & 0 & x_3 = 3 \\
-1 & 2 & 0 & 1 & 0 & x_4 = 8 \\
3 & 1 & 0 & 0 & 1 & x_5 = 18
\end{array}
$$

We use smallest index rule and run three iterations.

$$
\begin{array}{ccccc|c}
-1 & -3 & 0 & 0 & 0 & 0 \\ \hline
-1 & 1 & 1 & 0 & 0 & x_3 = 3 \\
-1 & 2 & 0 & 1 & 0 & x_4 = 8 \\
\boxed{3} & 1 & 0 & 0 & 1 & x_5 = 18
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & -\frac{8}{3} & 0 & 0 & \frac{1}{3} & 6 \\ \hline
0 & \frac{4}{3} & 1 & 0 & \frac{1}{3} & x_3 = 9 \\
0 & \boxed{\frac{7}{3}} & 0 & 1 & \frac{1}{3} & x_4 = 14 \\
1 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & x_1 = 6
\end{array}
$$

$$
\rightarrow
\begin{array}{ccccc|c}
0 & 0 & 0 & \frac{8}{7} & \frac{5}{7} & 22 \\ \hline
0 & 0 & 1 & -\frac{4}{7} & \frac{1}{7} & x_3 = 1 \\
0 & 1 & 0 & \frac{3}{7} & \frac{1}{7} & x_2 = 6 \\
1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & x_1 = 4
\end{array}
$$

an optimal solution to the LP is $(x_1^*, x_2^*) = (4, 6)$ with objective value $z^* = 22$. There isn't any iteration that has no improvement.

(d) We may think "highest-magnitude reduced cost" rule may have less iterations than smallest index rule, but we can know that it is not always true by Part (b) and (c).

5. The standard form is

$$
\begin{aligned}
\max \quad & x_1 + 3x_2 \\
\text{s.t.} \quad & -x_1 + x_2 + x_3 = 3 \\
& -x_1 + 2x_2 + x_4 = 8 \\
& 3x_1 + x_2 + x_5 = 18 \\
& x_i \geq 0 \quad \forall i = 1, ..., 5.
\end{aligned}
$$

(a)

$$
A_B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad
A_N = \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix} \quad
c_B = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad
c_N = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad
b = \begin{bmatrix} 3 \\ 8 \\ 18 \end{bmatrix}
$$

(b) The reduced costs are

$$\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T = \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 0 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & -2 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 3 \end{bmatrix}$$

→ We choose x_1 to enter because its reduced cost is the most negative among the nonbasic varibales.
→ $x_j = x_1$.

(c)

$$A_B^{-1} b = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 15 \end{bmatrix}$$

$$A_B^{-1} A_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$\rightarrow \text{ratio test: } \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{3}{-1} \\ \frac{2}{1} \\ \frac{15}{4} \end{bmatrix}$$

→ x_4 leaves.

(d) According (b) and (c), we now change our basis $B = (x_1, x_2, x_5)$. Then,

$$A_B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 2 & 0 \\ 3 & 1 & 1 \end{bmatrix} \quad A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad c_B = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \quad c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 8 \\ 18 \end{bmatrix}$$

We first find the entering variable as what we did in (b). The reduced costs are

$$\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ -1 & 1 & 0 \\ 7 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 7 & -4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 4 \end{bmatrix}$$

→ We choose x_3 to enter because its reduced cost is the most negative among the nonbasic varibales.
→ $x_j = x_3$.

Then, find the leaving variable as what we did in(c).

$$A_B^{-1}b = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 1 & 0 \\ 7 & -4 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 8 \\ 18 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$$

$$A_B^{-1}A_3 = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 1 & 0 \\ 7 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 7 \end{bmatrix}$$

$$\rightarrow \text{ratio test: } \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{2}{-2} \\ \frac{5}{-1} \\ \frac{7}{7} \end{bmatrix}$$

$\rightarrow x_5$ leaves.

The next basis B becomes (x_1, x_2, x_3).