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Suggested Solution for Homework 3

Solution providers: Kiwi Liu and Johnny Chen
Department of Information Management
National Taiwan University

1. (a) The gradient and Hessian are $[3x^2 + 2ax + b]$ and $[6x + 2a]$, respectively.
   (b) When $x \geq -\frac{a}{3}$.
   (c) When $x \leq -\frac{a}{3}$.
   (d) An optimal solution is $x^* = \frac{-a-\sqrt{a^2-3b}}{3}$. We let the $\nabla f(x^*)$ be 0. Notice that the result in Part (c) should be satisfied.

2. (a) We have $q^* = \sqrt{\frac{2KD}{h}}$ and $q' = \sqrt{\frac{KD}{h}}$.
   (b) We have $\frac{q'}{q^*} = \sqrt{\frac{KD}{h}\frac{h}{2KD}} = \sqrt{\frac{1}{2}}$.
   (c) We know $TC(q) = \frac{KD}{q} + \frac{hq}{2}$. Then $\frac{TC(q')}{TC(q*)} = \frac{\frac{3}{2}\sqrt{KDh}}{\sqrt{2KDH}} = \frac{3}{4}\sqrt{2}$.
   (d) New q' becomes $\sqrt{\frac{2KkD}{h}}$ Then, we have $\frac{TC(q')}{TC(q*)} = \frac{\frac{\sqrt{2}}{2}\sqrt{KDh}(\frac{1}{\sqrt{k}}+\sqrt{k})}{\sqrt{2KDh}} = \frac{1}{2}\frac{k+1}{\sqrt{k}}$.

3. (a) The function is convex if $x_2 \geq 0$ and $-4x_1^2 \geq 0$. As the result, the function is convex over the region $x_1 = 0, x_2 \geq 0$.
   (b) The function is nowhere convex.
   (c) The function is convex if $x_2 \geq 0$, $2x_2x_3 - x_1^2 \geq 0$, and $6x_2x_3^{-2} - 4x_2^3 - 3x_1^2x_3^{-3} \geq 0$
   (d) If $n = 2$ and 3, the Hessian matrix are

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } \nabla^2 f(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

   Follow the same rule, we know that no matter which number n is, $\nabla^2 f(x)$ is always positive semi-definite. Then, the function is convex everywhere.

4. (a) Let the decision variables be

$$q_i = \text{the sales quantity of product } i,\ i = 1, 2, 3,$$

   The seller's profit maximization problem is

$$\begin{aligned} \max \quad & f(q) = \sum_{i=1}^{3}(a - bq_i - c)q_i \\ \text{s.t.} \quad & \sum_{i=1}^{3} q_i \leq K \\ & q_i \geq 0 \quad \forall i = 1, 2, 3. \end{aligned}$$

   The Hessian matrix is

$$\nabla^2 f(q) = \begin{bmatrix} -2b & 0 & 0 \\ 0 & -2b & 0 \\ 0 & 0 & -2b \end{bmatrix}.$$

   Since $\nabla^2 f(q)$ is always less than 0. It is a concave function. And the constraint is a linear function. As the result, it is a convex program.

(b) The Lagrangian is $\mathcal{L}(q|\lambda) = \sum_{i=1}^{3}(a - 2bq_i - c)q_i + \lambda(K - \sum_{i=1}^{3} q_i)$
$\frac{\partial \mathcal{L}(q|\lambda)}{\partial q_i} = a - 2bq_i - c - \lambda \quad \forall i = 1, 2, 3$
The KKT condition for the problem is as follow ($\lambda \geq 0$):
   i. Primal feasibility: $\sum_{i=1}^{3} q_i \leq K$.
   ii. Dual feasibility: $a - 2bq_i - c - \lambda = 0 \quad \forall i = 1, 2, 3$.
   iii. Complementary slackness: $\lambda(K - \sum_{i=1}^{3} q_i) = 0$.

(c) By part(b), $q_i = \frac{a-c-\lambda}{2b} \quad \forall i = 1, 2, 3$. Because the constraint may be binding or nonbinding, there are two situation:
   i. If the constraint is binding, then $q_1+q_2+q_3 = K$. As the result $q_1 = q_2 = q_3 = \frac{a-c-\lambda}{2b} = \frac{K}{3}$
   ii. If the constraint is binding, then $q_i = \frac{a-c}{2b}$. Notice that the Lagrange multiplier λ is 0.

   Combine the above result, $q^* = \min\{\frac{a-c}{2b}, \frac{K}{3}\}$:
   i. $f(q^*) = aK - \frac{bK^2}{3} - cK$ when the constraint is binding.
   ii. $f(q^*) = \frac{(a-c)^2}{4b}$ when the constraint is nonbinding.

(d) The optimal quantity $q_i^* = \min\{\frac{a-c}{2b}, \frac{K}{3}\}$. Therefore it (weakly) increases in a, decreases in b, and decreases in c when the constraint is nonbinding. If the constraint is binding, the increasing of K will make q_i^* larger. The intuitive explanations are as below:
   i. For a, the reason is that the price is $a - bq_i$, thus increasing of a makes the product more profitable. The seller will want to sell more products.
   ii. For b, the reason is just the contrary of (i).
   iii. For c, the reason is that it is unit production cost. The increasing of c means that producing the product becomes more expensive.
   iv. For K, we know that $p_i - c$ must be greater than 0, otherwise the seller will not sell the product. The increasing of K means that the total demand becomes larger. As the shadow price of demand constraint is positive, selling products must be more profitable.

5. (a) The gradient and Hessian are

$$\nabla f(x) = \begin{bmatrix} e^{x_1} \\ 2x_2 \end{bmatrix} \text{ and } \nabla^2 f(x) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & 2 \end{bmatrix}.$$

(b) First, we set $x^0 = (2, 2)$ and the next solution be x^1.
We have

$$\nabla f(x^0) = \begin{bmatrix} e^2 \\ 4 \end{bmatrix} \text{ and } \nabla^2 f(x^0) = \begin{bmatrix} e^2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Therefore, we can obtain next solution by Newton's method:

$$x_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{1}{e^2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(c) The first step is the same as (b).
We have

$$\nabla f(x^0) = \begin{bmatrix} e^2 \\ 4 \end{bmatrix}.$$

Therefore, we can obtain next solution by the gradient descent method:

$$x_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} e^2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 - e^2 \\ -2 \end{bmatrix}.$$

(d) $a_0 = \text{argmin}_{a \geq 0} f(x^0 - a\nabla f(x^0))$,
where $f(x^0 - a\nabla f(x^0) = f(2 - ae^2, 2 - 4a) = e^{2-ae^2} + (2 - 4a)^2 = g(a)$.
By FOC, $g'(a) = -e^{4-ae^2} - 8(2 - 4a) = 0$ when $a \approx 0.533$ Therefore, we can obtain next solution by the gradient descent method:

$$x_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - 0.533 \begin{bmatrix} e^2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 - 0.533e^2 \\ -0.132 \end{bmatrix}.$$