Operations Research, Spring 2016
Suggested Solution for Pre-lecture Problems for Lecture 4

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1. (a) Its standard form is

$$
\begin{array}{rrcrcrcrcr}
\min & 5x_1 & + & 3x_2 & & & & & & \\
\text{s.t.} & x_1 & - & 4x_2 & + & x_3 & & & = & 4 \\
 & x_1 & & & & & + & x_4 & = & 8 \\
 & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.}
\end{array}
$$

$$
\begin{array}{cccc|c}
-5 & -3 & 0 & 0 & 0 \\ \hline
\boxed{1} & -4 & 1 & 0 & x_3 = 4 \\
1 & 0 & 0 & 1 & x_4 = 8
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & -23 & 5 & 0 & 20 \\ \hline
1 & -4 & 1 & 0 & x_3 = 4 \\
0 & \boxed{4} & -1 & 1 & x_4 = 4
\end{array}
$$

$$
\rightarrow
\begin{array}{cccc|c}
0 & 0 & -\frac{4}{3} & \frac{23}{4} & 43 \\ \hline
1 & 0 & 0 & 1 & x_1 = 8 \\
0 & 1 & -\frac{1}{4} & \frac{1}{4} & x_2 = 1
\end{array}
$$

Since there is no positive denominators in the ratio test, this LP is unbounded.

(b)

$$
\begin{array}{cccc|c}
-5 & -3 & 0 & 0 & 0 \\ \hline
1 & -4 & 1 & 0 & x_3 = 4 \\
1 & 0 & 0 & 1 & x_4 = 8
\end{array}
$$

Since there is no positive denominators in the ratio test, this LP is unbounded.

(c) The search routes of the two above solution processes is shown in Figure 1.

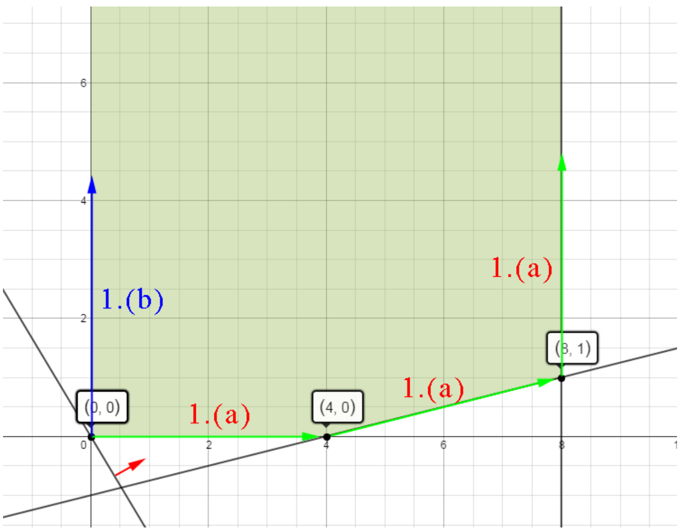

Figure 1: Search routes for Problem 1

2. (a) Its standard form is

$$\begin{array}{llllllll}
\max & 5x_1 & + & 3x_2 & & & & \\
\text{s.t.} & x_1 & + & x_2 & + x_3 & & & = 12 \\
 & x_1 & + & 4x_2 & & + x_4 & & = 20 \\
 & & & x_2 & & & - x_5 & = 8 \\
 & x_i \geq 0 & \forall i = 1, ..., 5. & & & & &
\end{array}$$

(b) Its Phase-I LP is

$$\begin{array}{lllllllll}
\min & & & & & & & x_6 & \\
\text{s.t.} & x_1 & + & x_2 & + x_3 & & & & = 12 \\
 & x_1 & + & 4x_2 & & + x_4 & & & = 20 \\
 & & & x_2 & & & - x_5 & + x_6 & = 8 \\
 & x_i \geq 0 & \forall i = 1, ..., 6. & & & & & &
\end{array}$$

(c)

$$\begin{array}{cccccc|c}
0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline
1 & 1 & 1 & 0 & 0 & 0 & x_3 = 12 \\
1 & 4 & 0 & 1 & 0 & 0 & x_4 = 20 \\
0 & 1 & 0 & 0 & -1 & 1 & x_6 = 8
\end{array}
\quad \overset{\text{adjust}}{\rightarrow} \quad
\begin{array}{cccccc|c}
0 & 1 & 0 & 0 & -1 & 0 & 8 \\ \hline
1 & 1 & 1 & 0 & 0 & 0 & x_3 = 12 \\
1 & \boxed{4} & 0 & 1 & 0 & 0 & x_4 = 20 \\
0 & 1 & 0 & 0 & -1 & 1 & x_6 = 8
\end{array}$$

$$\rightarrow \quad
\begin{array}{cccccc|c}
-\frac{1}{4} & 0 & 0 & -\frac{1}{4} & -1 & 0 & 3 \\ \hline
\frac{3}{4} & 0 & 1 & -\frac{1}{4} & 0 & 0 & x_3 = 7 \\
\frac{1}{4} & 1 & 0 & \frac{1}{4} & 0 & 0 & x_2 = 8 \\
-\frac{1}{4} & 0 & 0 & -\frac{1}{4} & -1 & 1 & x_6 = 3
\end{array}$$

In the Phase-I optimal solution $(0, 8, 7, 0, 0, 3)$, the artificial variable x_6 is still in the basis and positive. Therefore, we conclude that the original LP is infeasible.

3. (a) Its Phase-I LP is

$$\begin{array}{llllllllll}
\min & & & & & & & x_6 & + x_7 & \\
\text{s.t.} & x_1 & + & x_2 & + x_3 & & & & & = 12 \\
 & x_1 & + & 4x_2 & & - x_4 & & + x_6 & & = 20 \\
 & & & x_2 & & & - x_5 & & + x_7 & = 8 \\
 & x_i \geq 0 & \forall i = 1, ..., 7. & & & & & & &
\end{array}$$

$$\begin{array}{ccccccc|c}
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & x_3 = 12 \\
1 & 4 & 0 & -1 & 0 & 1 & 0 & x_6 = 20 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & x_7 = 8
\end{array}
\quad \overset{\text{adjust}}{\rightarrow} \quad
\begin{array}{ccccccc|c}
1 & 5 & 0 & -1 & -1 & 0 & 0 & 28 \\ \hline
\boxed{1} & 1 & 1 & 0 & 0 & 0 & 0 & x_3 = 12 \\
1 & 4 & 0 & -1 & 0 & 1 & 0 & x_6 = 20 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & x_7 = 8
\end{array}$$

$$\rightarrow \quad
\begin{array}{ccccccc|c}
0 & 4 & -1 & -1 & -1 & 0 & 0 & 16 \\ \hline
1 & 1 & 1 & 0 & 0 & 0 & 0 & x_1 = 12 \\
0 & \boxed{3} & -1 & -1 & 0 & 1 & 0 & x_6 = 8 \\
0 & 1 & 0 & 0 & -1 & 0 & 1 & x_7 = 8
\end{array}
\quad \rightarrow \quad
\begin{array}{cccccc|c}
0 & 0 & \frac{1}{3} & \frac{1}{3} & -1 & 0 & \frac{16}{3} \\ \hline
1 & 0 & \boxed{\frac{4}{3}} & \frac{1}{3} & 0 & 0 & x_1 = \frac{28}{3} \\
0 & 1 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & x_2 = \frac{8}{3} \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & -1 & 1 & x_7 = \frac{16}{3}
\end{array}$$

$$\rightarrow \quad
\begin{array}{cccccc|c}
-\frac{1}{4} & 0 & 0 & \frac{1}{4} & -1 & 0 & 3 \\ \hline
\frac{3}{4} & 0 & 1 & \frac{1}{4} & 0 & 0 & x_3 = 7 \\
\frac{1}{4} & 1 & 0 & -\frac{1}{4} & 0 & 0 & x_2 = 5 \\
-\frac{1}{4} & 0 & 0 & \boxed{\frac{1}{4}} & -1 & 1 & x_7 = 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
0 & 0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & 1 & 0 & 1 & x_3 = 4 \\
0 & 1 & 0 & 0 & -1 & x_2 = 8 \\
-1 & 0 & 0 & 1 & -4 & x_4 = 12
\end{array}$$

We found a feasible solution $(0, 8, 4, 12, 0)$.

(b)

$$\begin{array}{ccccc|c} -5 & -3 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & x_3 = 4 \\ 0 & 1 & 0 & 0 & -1 & x_2 = 8 \\ -1 & 0 & 0 & 1 & -4 & x_4 = 12 \end{array} \underset{\rightarrow}{\overset{\text{adjust}}{\overbrace{}}} \begin{array}{ccccc|c} -5 & 0 & 0 & 0 & -3 & 24 \\ \hline \boxed{1} & 0 & 1 & 0 & 1 & x_3 = 4 \\ 0 & 1 & 0 & 0 & -1 & x_2 = 8 \\ -1 & 0 & 0 & 1 & -4 & x_4 = 12 \end{array}$$

$$\rightarrow \begin{array}{ccccc|c} 0 & 0 & 5 & 0 & 2 & 44 \\ \hline 1 & 0 & 1 & 0 & 1 & x_1 = 4 \\ 0 & 1 & 0 & 0 & -1 & x_2 = 8 \\ 0 & 0 & 1 & 1 & -3 & x_4 = 16 \end{array}$$

We found bfs $x^* = (4, 8, 0, 16, 0)$.