Operations Research, Spring 2017
Suggested Solution For Homework 1

Solution providers: Peter Chien and Jeremy Chang
Department of Information Management
National Taiwan University

1. (a) The solution is shown in Figure 1.

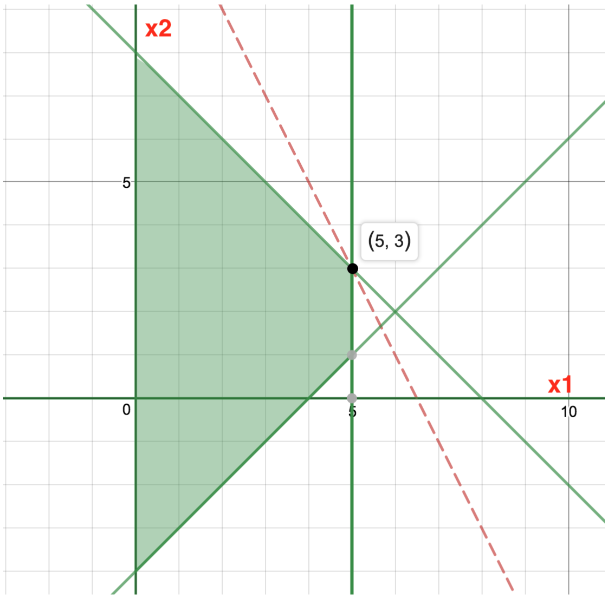

Figure 1: Graphical solution for Problem 1.(a)

According to Figure 1, the optimal solution is $(x_1^*, x_2^*) = (5, 3)$, and the optimal objective value is $z^* = 13$.

(b) The basic feasible solutions are

basis	x_1	x_2	x_3	x_4	x_5	x_6
$\{x_1, x_2, x_4\}$	5	3	0	2	0	0
$\{x_1, x_2, x_5\}$	5	1	0	0	2	0
$\{x_1, x_5, x_6\}$	4	0	0	0	4	1
$\{x_2, x_4, x_6\}$	0	8	0	12	0	5
$\{x_3, x_5, x_6\}$	0	0	4	0	12	5
$\{x_4, x_5, x_6\}$	0	0	0	4	8	5

(c) The standard form is

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & 2x_1 & + & x_2 & - & x_3 & & & & & & & & \\
\text{s.t.} & x_1 & - & x_2 & + & x_3 & + & x_4 & & & & & = & 4 \\
 & x_1 & + & x_2 & - & x_3 & & & + & x_5 & & & = & 8 \\
 & x_1 & & & & & & & & & + & x_6 & = & 5 \\
 & \multicolumn{13}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6.}
\end{array}
$$

The problem solving processes are

$$\begin{array}{cccccc|c} -2 & -1 & 1 & 0 & 0 & 0 & 0 \\ \hline \boxed{1} & -1 & 1 & 1 & 0 & 0 & x_4=4 \\ 1 & 1 & -1 & 0 & 1 & 0 & x_5=8 \\ 1 & 0 & 0 & 0 & 0 & 1 & x_6=5 \end{array} \rightarrow \begin{array}{cccccc|c} 0 & -3 & 3 & 2 & 0 & 0 & 8 \\ \hline 1 & -1 & 1 & 1 & 0 & 0 & x_1=4 \\ 0 & 2 & -2 & -1 & 1 & 0 & x_5=4 \\ 0 & \boxed{1} & -1 & -1 & 0 & 1 & x_6=1 \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} 0 & 0 & 0 & -1 & 0 & 3 & 11 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & x_1=5 \\ 0 & 0 & 0 & \boxed{1} & 1 & -2 & x_5=2 \\ 0 & 1 & -1 & -1 & 0 & 1 & x_2=1 \end{array} \rightarrow \begin{array}{cccccc|c} 0 & 0 & 0 & 0 & 1 & 1 & 13 \\ \hline 1 & 0 & 0 & 0 & 0 & 1 & x_1=5 \\ 0 & 0 & 0 & 1 & 1 & -2 & x_4=2 \\ 0 & 1 & -1 & 0 & 1 & -1 & x_2=3 \end{array}$$

The optimal solution is $(x_1^*, x_2^*) = (5, 3)$ and the optimal objective value is $z^* = 13$.

(d) The standard form is

$$\begin{array}{rcccccccccccccc} \max & x_1 & - & 2x_2 & + & 2x_3 & & & & & & & & \\ \text{s.t.} & x_1 & - & x_2 & + & x_3 & + & x_4 & & & & & = & 4 \\ & x_1 & + & x_2 & - & x_3 & & & + & x_5 & & & = & 8 \\ & x_1 & & & & & & & & & + & x_6 & = & 5 \\ & \multicolumn{14}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6.} \end{array}$$

$$\begin{array}{cccccc|c} -1 & 2 & -2 & 0 & 0 & 0 & 0 \\ \hline \boxed{1} & -1 & 1 & 1 & 0 & 0 & x_4=4 \\ 1 & 1 & -1 & 0 & 1 & 0 & x_5=8 \\ 1 & 0 & 0 & 0 & 0 & 1 & x_6=5 \end{array} \rightarrow \begin{array}{cccccc|c} 0 & 1 & -1 & 1 & 0 & 0 & 4 \\ \hline 1 & -1 & \boxed{1} & 1 & 0 & 0 & x_1=4 \\ 0 & 2 & -2 & -1 & 1 & 0 & x_5=4 \\ 0 & 1 & -1 & -1 & 0 & 1 & x_6=1 \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} 1 & 0 & 0 & 2 & 0 & 0 & 8 \\ \hline 1 & -1 & 1 & 1 & 0 & 0 & x_3=4 \\ 2 & 0 & 0 & 1 & 1 & 0 & x_5=12 \\ 1 & 0 & 0 & 0 & 0 & 1 & x_6=5 \end{array}$$

The optimal solution is $(x_1^*, x_2^*) = (0, -4)$ and the optimal objective value is $z^* = 8$..

2. (a) The standard form of Phase-I LP is

$$\begin{array}{rcccccccccccccccc} \max & & & & & & & & & & & & x_7 & + & x_8 & & \\ \text{s.t.} & x_1 & + & x_2 & & & + & x_4 & & & & & & & & = & 4 \\ & x_1 & + & 2x_2 & + & x_3 & & & - & x_5 & & & + & x_7 & & = & 10 \\ & & & & & x_3 & & & & & - & x_6 & & + & x_8 & = & 3 \\ & \multicolumn{16}{l}{x_i \geq 0 \quad \forall i = 1, ..., 8.} \end{array}$$

The problem solving processes of Phase-I LP are

$$\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x_4=4 \\ 1 & 2 & 3 & 0 & -1 & 0 & 1 & 0 & x_7=10 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & x_8=3 \end{array} \underbrace{\text{adjust}}_{\rightarrow} \begin{array}{cccccccc|c} 1 & 2 & 4 & 0 & -1 & -1 & 0 & 0 & 13 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x_4=4 \\ 1 & 2 & 3 & 0 & -1 & 0 & 1 & 0 & x_7=10 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 0 & 1 & x_8=3 \end{array}$$

$$\rightarrow\ \begin{array}{cccccccc|c}
1 & 2 & 0 & 0 & -1 & 3 & 0 & -4 & 1 \\ \hline
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & x_4 = 4 \\
1 & \boxed{2} & 0 & 0 & -1 & 3 & 1 & -3 & x_7 = 1 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & x_3 = 3
\end{array}
\ \rightarrow\ \begin{array}{cccccccc|c}
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline
\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} & x_4 = \frac{7}{2} \\
\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} & x_2 = \frac{1}{2} \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & x_3 = 3
\end{array}$$

The problem solving processes of Phase-II LP are

$$\begin{array}{cccccc|c}
-1 & -3 & -2 & 0 & 0 & 0 & 0 \\ \hline
\frac{1}{2} & 0 & 0 & 1 & \frac{1}{2} & -\frac{3}{2} & x_4 = \frac{7}{2} \\
\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & x_2 = \frac{1}{2} \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}
\ \underbrace{\rightarrow}_{\text{adjust}}\ \begin{array}{cccccc|c}
\frac{1}{2} & 0 & 0 & 0 & -\frac{3}{2} & \frac{5}{2} & \frac{15}{2} \\ \hline
\frac{1}{2} & 0 & 0 & 1 & \boxed{\frac{1}{2}} & -\frac{3}{2} & x_4 = \frac{7}{2} \\
\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & x_2 = \frac{1}{2} \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}$$

$$\rightarrow\ \begin{array}{cccccc|c}
2 & 0 & 0 & 3 & 0 & -2 & 18 \\ \hline
1 & 0 & 0 & 2 & 1 & -3 & x_5 = 7 \\
1 & 1 & 0 & 1 & 0 & 0 & x_2 = 4 \\
0 & 0 & 1 & 0 & 0 & -1 & x_3 = 3
\end{array}$$

In column 6, the reduce cost is negative and all numbers in row 1 to 3 are nonpositive. Therefore, the LP is unbounded.

(b) The model file is shown in Figure 2 and result is in Figure 3.

```
var x1;
var x2;
var x3;

    maximize profit: x1 +3*x2 +2*x3;

    subject to resource_1: x1+x2 <= 4;
    subject to resource_2: x1 +2*x2 +3*x3 >= 10;
    subject to resource_3: x3 >= 3;

    subject to nonneg_1: x1 >= 0;
    subject to nonneg_2: x2 >= 0;
    subject to nonneg_3: x3 >= 0;
```

Figure 2: Graphical solution for Problem 2.(b)

```
CPLEX 12.7.0.0: unbounded problem.
1 dual simplex iterations (1 in phase I)
variable.unbdd returned

suffix unbdd OUT;
```

Figure 3: Graphical solution for Problem 2.(b)

3. (a) false. The counterexample is

$$\begin{array}{rcccl}
\min & x_1 & + & x_2 & \\
\text{s.t.} & x_1 & + & x_2 & \geq \ 5
\end{array}$$

$$x_i \geq 0 \quad \forall i = 1, ..., 2.$$

In the counterexample, an LPs optimal solution is not always an extreme point.

(b) true. The LP has fewer than $\binom{n}{m}$ bases when two of constraints are parallel.

(c) true
You can increase the variable according to the column without limitation in order to lower the objective value.

(d) false. We can only guarantee to solve a minimization LP in finite step.There is no variable selection rule can guarantee to generate the least iteration.

4. (a) We label Monday as day 1, Tuesday as day 2, Wednesday as day 3, etc.

Let the parameters be

$$D_{ij} = \text{number of students needed for slot } i \text{ in day } j.$$

Let the decision variables be

$$x_{jk} = \text{number of students work on day } j \text{ and day } k,\ k{>}j.$$

$$\begin{aligned}
\min \quad & \sum_{j=1}^{4}\sum_{k=j+1}^{5} x_{jk} \\
\text{s.t.} \quad & x_{12}+x_{13}+x_{14}+x_{15} \geq D_{i1} \quad \forall i=1,...,8 \\
& x_{12}+x_{23}+x_{24}+x_{25} \geq D_{i2} \quad \forall i=1,...,8 \\
& x_{13}+x_{23}+x_{34}+x_{35} \geq D_{i3} \quad \forall i=1,...,8 \\
& x_{14}+x_{24}+x_{34}+x_{45} \geq D_{i4} \quad \forall i=1,...,8 \\
& x_{15}+x_{25}+x_{35}+x_{45} \geq D_{i5} \quad \forall i=1,...,8 \\
& x_{jk} \geq 0 \quad \forall j=1,...,4 \quad k=2,...,5 \quad k>j.
\end{aligned}$$

(b) Let the parameters be

$$\begin{aligned}
& D_{ij} = \text{number of students that are needed for slot } i \text{ in day } j, \\
& N = \text{number of cadidates that can be assigned to work.}
\end{aligned}$$

Let the decision variables be

$$\begin{aligned}
& x_{1jk} = 1 \text{ if student } k \text{ works from slot 1 to 4 on day } j \text{ or 0 otherwise.} \\
& x_{5jk} = 1 \text{ if student } k \text{ works from slot 5 to 8 on day } j \text{ or 0 otherwise.} \\
& z_k = (\text{number of student } k\text{'s working slots}) \ / \ 16.
\end{aligned}$$

$$\begin{aligned}
\min \quad & \sum_{k=1}^{N} z_k \\
\text{s.t.} \quad & \sum_{k=1}^{N} x_{ijk} \geq \max\{D_{ij}, D_{i+1,j}, D_{i+2,j}, D_{i+3,j}\} \quad \forall i=1,5 \quad \forall j=1,...,5. \\
& \sum_{j=1}^{5} (x_{1jk}+x_{5jk}) \leq 16 z_k \quad \forall k=1,..,N \\
& x_{ijk} \leq 1 \quad \forall i=1,5 \quad \forall j=1,...,5 \quad \forall k=1,..,N \\
& x_{ijk} \geq 0 \quad \forall i=1,5 \quad \forall j=1,...,5 \quad \forall k=1,..,N \\
& z_k \leq 1 \quad \forall k=1,..,N \\
& z_k \geq 0 \quad \forall k=1,..,N
\end{aligned}$$

(c) Figure 4 shows the AMPL model file. Figure 5 shows the AMPL data file. Figure 6 is the result. Figure 7 and Figure 8 is the initial schedule.

We assume there are twenty candidates that works for STIM. In this LP program, the optimal objective value is 17.25. That is, we need at least 18 students to work for IM Week. The following steps are our adjustment procedures:

- Assign x_{114} to student 9.
- Assign x_{117} to student 10.
- Assign x_{124} to student 16.
- Assign x_{537} to student 17.

In the shift, we focus on slots and day instead of individual student so we use x_{ij} instead of x_{ijk}. Thus, the schedule is:

- 1 people: x_{11}, x_{51}, x_{54}, x_{55}.
- 2 people: x_{11}, x_{12}, x_{52}, x_{14}.
- 1 people: x_{11}, x_{12}, x_{53}.
- 1 people: x_{11}, x_{12}, x_{14}, x_{54}.
- 1 people: x_{11}, x_{13}, x_{53}, x_{54}.
- 1 people: x_{11}, x_{53}, x_{54}, x_{55}.
- 1 people: x_{11}, x_{53}, x_{55}.
- 1 people: x_{51}, x_{52}, x_{53}, x_{54}.
- 1 people: x_{51}, x_{52}, x_{14}, x_{55}.
- 1 people: x_{51}, x_{52}, x_{54}, x_{55}.
- 1 people: x_{51}, x_{13}, x_{14}, x_{15}.
- 1 people: x_{51}, x_{53}, x_{55}.
- 1 people: x_{51}, x_{14}, x_{54}, x_{55}.
- 1 people: x_{51}, x_{54}, x_{15}, x_{55}.
- 1 people: x_{12}, x_{52}, x_{14}, x_{15}.
- 2 people: x_{12}, x_{13}, x_{14}, x_{15}.

STIM needs 18 students in total.

```
param S; #8
param D; #5
param N; #number of cadidates that can be assigned to work.
param DayNeeded{l in 1..S, j in 1..D};

var x{i in 1..2, j in 1..D, k in 1..N};
var z{k in 1..N}; #(number of student k's working slots) / 16.

minimize profit: sum{k in 1..N}(z[k]);

subject to slotConstraint1{i in 1..2,j in 1..D, l in 4*i-3..4*i}:
    sum{k in 1..N}(x[i,j,k]) >=DayNeeded[l, j];
subject to maxConstraint{k in 1..N}:
    4*z[k]>=sum{j in 1..5}(x[1,j,k]+x[2,j,k]);
subject to binaryConstraint1{i in 1..2, j in 1..D, k in 1..N}:
    x[i,j,k] <=1;
subject to binaryConstraint2{k in 1..N}:
    z[k] <=1;
subject to nonnegX{i in 1..2, j in 1..D, k in 1..N}:
    x[i, j, k] >=0;
subject to nonnegZ{k in 1..N}:
    z[k] >=0;
```

Figure 4: AMPL model for problem 4.(c)

```
param S := 8;
param D := 5;
param N = 20;
param DayNeeded: 1 2 3 4 5 :=
1 6 4 4 5 4
2 3 2 1 3 4
3 3 2 1 4 3
4 8 7 4 9 5
5 8 6 6 8 3
6 4 2 2 2 3
7 3 2 2 2 5
8 2 1 1 2 8;
```

Figure 5: AMPL data for problem 4.(c)

```
sw: ampl
ampl: option solver cplex;
ampl: model eg3.mod;
ampl: data eg3.dat;
ampl: solve;
CPLEX 12.6.3.0: optimal solution; objective 17.25
69 dual simplex iterations (1 in phase I)
ampl: display z;
z [*] :=
 1  1
 2  1
 3  1
 4  0.5
 5  0.75
 6  0.75
 7  0.5
 8  0.75
 9  0.75
10  0.75
11  1
12  1
13  1
14  1
15  1
16  0.75
17  0.75
18  1
19  1
20  1
;
```

Figure 6: AMPL solution for problem 4.(c)

```
ampl: display x;
x [1,*,*] (tr)
:     1    2    3    4    5     :=
1     0    1    0    1    1
2     1    1    0    1    0
3     1    1    0    1    0
4     1    1    0    0    0
5     1    1    0    0    0
6     1    0    0    0    0
7     1    0    0    0    0
8     0    0    0    0    0
9     0    0    0    0    0
10    0    0    0    0    0
11    0    0    0    0    1
12    0    0    0    0    0
13    0    0    1    1    1
14    0    1    1    1    1
15    0    1    1    1    1
16    1    0    0    1    0
17    1    0    1    0    0
18    0    0    0    1    0
19    0    0    0    0    0
20    0    0    0    1    0
```

Figure 7: AMPL shift result for problem 4.(c) (1)

```
 [2,*,*] (tr)
:     1    2    3    4    5      :=
1     0    1    0    0    0
2     0    1    0    0    0
3     0    1    0    0    0
4     0    0    0    0    0
5     0    0    1    0    0
6     0    0    1    0    1
7     0    0    1    0    0
8     1    0    1    0    1
9     0    0    1    1    1
10    1    0    0    1    1
11    1    0    0    1    1
12    1    1    1    1    0
13    1    0    0    0    0
14    0    0    0    0    0
15    0    0    0    0    0
16    0    0    0    1    0
17    0    0    0    1    0
18    1    0    0    1    1
19    1    1    0    1    1
20    1    1    0    0    1
;
```

Figure 8: AMPL shift result for problem 4.(c) (2)