Operations Research, Spring 2017
Suggested Solution for Pre-lecture Problems for Lecture 5

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1. The dual LP is

$$\begin{array}{rrcrcrcr} \min & 10y_1 & + & 16y_2 & + & 14y_3 & & \\ \text{s.t.} & 2y_1 & & & + & y_3 & \geq & 4 \\ & y_1 & + & y_2 & + & 3y_3 & \leq & -2 \\ & & & y_2 & - & 3y_3 & = & 1 \\ & \multicolumn{7}{l}{y_1 \geq 0, \quad y_2 \leq 0, \quad y_3 \quad urs.} \end{array}$$

2. (a) Its standard form is

$$\begin{array}{rrcrcrcrcr} \min & 3x_1 & + & 5x_2 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & = & 8 \\ & x_1 & + & 2x_2 & & & + & x_4 & = & 12 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$

$$\begin{array}{rrrr|c} -3 & -5 & 0 & 0 & 0 \\ \hline \boxed{1} & 1 & 1 & 0 & x_3 = 8 \\ 1 & 2 & 0 & 1 & x_4 = 12 \end{array} \rightarrow \begin{array}{rrrr|c} 0 & -2 & 3 & 0 & 24 \\ \hline 1 & 1 & 1 & 0 & x_1 = 8 \\ 0 & \boxed{1} & -1 & 1 & x_4 = 4 \end{array}$$

$$\rightarrow \begin{array}{rrrr|c} 0 & 0 & 1 & 2 & 32 \\ \hline 1 & 0 & 2 & -1 & x_1 = 4 \\ 0 & 1 & -1 & 1 & x_2 = 4 \end{array}$$

The optimal solution is $\bar{x} = (4, 4)$ with an objective value 32.

(b) The dual LP is

$$\begin{array}{rrcrcr} \min & 8y_1 & + & 12y_2 & & \\ \text{s.t.} & y_1 & + & y_2 & \geq & 3 \\ & y_1 & + & 2y_2 & \geq & 5 \\ & \multicolumn{5}{l}{y_i \geq 0 \quad \forall i = 1, 2.} \end{array}$$

(c) In Phase-I we get

$$\begin{array}{rrcrcrcrcrcrcr} \min & & & & & & & & & y_5 & + & y_6 & & \\ \text{s.t.} & y_1 & + & y_2 & - & y_3 & & & + & y_5 & & & = & 3 \\ & y_1 & + & 2y_2 & & & - & y_4 & & & + & y_6 & = & 5 \\ & \multicolumn{13}{l}{y_i \geq 0 \quad \forall i = 1, ..., 6.} \end{array}$$

$$\begin{array}{rrrrrr|c} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 1 & -1 & 0 & 1 & 0 & y_5 = 3 \\ 1 & 2 & 0 & -1 & 0 & 1 & y_6 = 5 \end{array} \overset{\text{adjust}}{\overbrace{\rightarrow}} \begin{array}{rrrrrr|c} 2 & 3 & -1 & -1 & 0 & 0 & 8 \\ \hline \boxed{1} & 1 & -1 & 0 & 1 & 0 & y_5 = 3 \\ 1 & 2 & 0 & -1 & 0 & 1 & y_6 = 5 \end{array}$$

$$\rightarrow \begin{array}{rrrr|c} 0 & 1 & 1 & -1 & 2 \\ \hline 1 & 1 & -1 & 0 & y_1 = 3 \\ 0 & \boxed{1} & 1 & 1 & y_6 = 2 \end{array} \rightarrow \begin{array}{rrrr|c} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -2 & 1 & y_1 = 1 \\ 0 & 1 & 1 & -1 & y_2 = 2 \end{array}$$

In Phase-II we get

$$\begin{array}{cccc|c} -8 & -12 & 0 & 0 & 32 \\ \hline 1 & 0 & -2 & 1 & y_1 = 1 \\ 0 & 1 & 1 & -1 & y_2 = 2 \end{array}$$

The optimal solution is $\bar{y} = (1, 2)$ with an objective value 32.

$$c^T\bar{x} = \left[\begin{array}{cc} 3 & 5 \end{array}\right] \left[\begin{array}{c} 4 \\ 4 \end{array}\right] = 32 = \left[\begin{array}{cc} 1 & 2 \end{array}\right] \left[\begin{array}{c} 8 \\ 12 \end{array}\right] = \bar{y}^T b.$$

3. (a) Its standard form is

$$\begin{array}{rrcrcrcrcl} \max & 5x_1 & + & 3x_2 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & = & 12 \\ & x_1 & + & 2x_2 & & & + & x_4 & = & 16 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$

$$\begin{array}{cccc|c} -5 & -3 & 0 & 0 & 0 \\ \hline \boxed{1} & 1 & 1 & 0 & x_3 = 12 \\ 1 & 2 & 0 & 1 & x_4 = 16 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} 0 & 2 & 5 & 0 & 60 \\ \hline 1 & 1 & 1 & 0 & x_1 = 12 \\ 0 & 1 & -1 & 1 & x_4 = 4 \end{array}$$

The optimal solution is $\bar{x} = (12, 0)$ with an objective value 60.

(b) The dual LP is

$$\begin{array}{rrcrcl} \min & 12y_1 & + & 16y_2 & & \\ \text{s.t.} & y_1 & + & y_2 & \geq & 5 \\ & y_1 & + & 2y_2 & \geq & 3 \\ & \multicolumn{5}{l}{y_i \geq 0 \quad \forall i = 1, 2.} \end{array}$$

(c) In Phase-I we get

$$\begin{array}{rrcrcrcrcrcrcl} \min & & & & & & & & & y_5 & + & y_6 & & \\ \text{s.t.} & y_1 & + & y_2 & - & y_3 & & & + & y_5 & & & = & 5 \\ & y_1 & + & 2y_2 & & & - & y_4 & & & + & y_6 & = & 3 \\ & \multicolumn{13}{l}{y_i \geq 0 \quad \forall i = 1, ..., 6.} \end{array}$$

$$\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 1 & -1 & 0 & 1 & 0 & y_5 = 5 \\ 1 & 2 & 0 & -1 & 0 & 1 & y_6 = 3 \end{array} \quad \overbrace{\xrightarrow{}}^{\text{adjust}} \quad \begin{array}{cccccc|c} 2 & 3 & -1 & -1 & 0 & 0 & 8 \\ \hline 1 & 1 & -1 & 0 & 1 & 0 & y_5 = 5 \\ \boxed{1} & 2 & 0 & -1 & 0 & 1 & y_6 = 3 \end{array}$$

$$\rightarrow \quad \begin{array}{cccc|c} 0 & -1 & -1 & 1 & 2 \\ \hline 0 & -1 & -1 & \boxed{1} & y_5 = 2 \\ 1 & 2 & 0 & -1 & y_1 = 3 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & 1 & y_4 = 2 \\ 1 & 1 & -1 & 0 & y_1 = 5 \end{array}$$

In Phase-II we get

$$\begin{array}{cccc|c} -12 & -16 & 0 & 0 & 60 \\ \hline 0 & -1 & -1 & 1 & y_4 = 2 \\ 1 & 1 & -1 & 0 & y_1 = 5 \end{array}$$

The optimal solution is $\bar{y} = (5, 0)$ with an objective value 60.

$$c^T\bar{x} = \left[\begin{array}{cc} 5 & 3 \end{array}\right] \left[\begin{array}{c} 12 \\ 0 \end{array}\right] = 60 = \left[\begin{array}{cc} 5 & 0 \end{array}\right] \left[\begin{array}{c} 12 \\ 16 \end{array}\right] = \bar{y}^T b.$$

(d)

$$c_B^T A_B^{-1} = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix} = \bar{y}^T.$$

(e) By proposition 8, for any LP, shadow prices equal the values of dual variables in the dual optimal solution. Therefore, the shadow price for the first and the second primal constraints are 5 and 0, respectively.