Operations Research, Spring 2014
Suggested Solution for Midterm Exam

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1. (a) The standard form in phase I is

$$\begin{array}{ll} \min & x_4 \\ \text{s.t.} & x_1' - x_1'' + 2x_2 + x_3 = 4 \\ & x_1' - x_1'' + x_2 + x_4 = 3 \\ & x_1', x_1'', x_2, x_3, x_4 \geq 0. \end{array}$$

The iterations for phase I are as follows:

$$\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 1 & -1 & 2 & 1 & 0 & 4\ (x_3) \\ 1 & -1 & 1 & 0 & 1 & 3\ (x_4) \end{array} \rightarrow \begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 3 \\ \hline 1 & -1 & 2 & 1 & 0 & 4\ (x_3) \\ \boxed{1} & -1 & 1 & 0 & 1 & 3\ (x_4) \end{array}$$

$$\rightarrow \begin{array}{ccccc|c} 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 1 & 1 & -1 & 1\ (x_3) \\ 1 & -1 & 1 & 0 & 1 & 3\ (x_1') \end{array}.$$

(b) The standard form in phase II is

$$\begin{array}{ll} \max & 2x_1' - 2x_1'' + 3x_2 \\ \text{s.t.} & x_1' - x_1'' + 2x_2 + x_3 = 4 \\ & x_1' - x_1'' + x_2 + x_4 = 3 \\ & x_1', x_1'', x_2, x_3, x_4 \geq 0. \end{array}$$

The iterations for phase II are as follows:

$$\begin{array}{cccc|c} -2 & 2 & -3 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1\ (x_3) \\ 1 & -1 & 1 & 0 & 3\ (x_1') \end{array} \rightarrow \begin{array}{cccc|c} 0 & 0 & -1 & 0 & 6 \\ \hline 0 & 0 & \boxed{1} & 1 & 1\ (x_3) \\ 1 & -1 & 1 & 0 & 3\ (x_1') \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 0 & 0 & 0 & 1 & 7 \\ \hline 0 & 0 & 1 & 1 & 1\ (x_2) \\ 1 & -1 & 0 & -1 & 2\ (x_1') \end{array}.$$

An optimal solution to the original LP is $(x_1, x_2) = (2, 1)$ with objective value $z^* = 7$.

2. (a) The maximum number of edges that may be selected in an n-node complete graph is $\lfloor \frac{n}{2} \rfloor$.

(b) Let the decision variables be

$$x_{ij} = \begin{cases} 1 & \text{if the edge between node } i \text{ and node } j \text{ is selected} \\ 0 & \text{otherwise} \end{cases}, [i,j] \in E.$$

The linear integer formulation is

$$\begin{array}{ll} \max & \sum_{[i,j] \in E} x_{ij} \\ \text{s.t.} & \sum_{[k,i] \in E} x_{ki} + \sum_{[i,j] \in E} x_{ij} \leq 1 \quad \forall i \in V \\ & x_{ij} \in \{0, 1\} \quad \forall [i,j] \in E. \end{array}$$

3. (a) We formulate the problem as a transportation problem by making each worker a supply node and each job a demand node. Each supply node has a supply quantity 2 and each demand node has a demand quantity 1. Moreover, we add a virtual demand node, node 6, with demand quantity 5. The cost between a supply node i and a demand node j for $j = 1, ..., 5$ is c_{ij} for $i = 1, ..., 5$. For demand node 6, the cost from a supply node i to it is $c_{i,6} = 0$ for $i = 1, ..., 5$. With this setting, we may then use a transportation solver to find the minimum-cost way for shipping items from the supply nodes to the demand nodes. It then implies how to assign jobs to workers with the minimum cost.

(b) Let the decision variables be

$$x_{ij} = \text{the number of job } j \text{ that worker } i \text{ does}, \ i = 1, ..., 5, j = 1, ..., 6.$$

The complete formulation is

$$\begin{aligned}
\min \quad & \sum_{i=1}^{5}\sum_{j=1}^{6} c_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{6} x_{ij} = 2 \quad \forall i = 1, ..., 5 \\
& \sum_{i=1}^{5} x_{ij} = 1 \quad \forall j = 1, ..., 5 \\
& \sum_{i=1}^{5} x_{i6} = 5 \\
& x_{ij} \in \mathbb{Z}_+ \quad \forall i = 1, ..., 5, j = 1, ..., 6.
\end{aligned}$$

The objective function is to minimize the total cost, and those three constraints guarantee the input is equal to the output for each node. The coefficient matrix is totally unimodular. If you divide the constraints by putting the first constraint in one group and putting the second and third constraint in the other group, then for each column two nonzero elements will not be in the same group. Besides, all the elements are either 1, 0, or -1, and each column contains at most two nonzero elements. Therefore, the coefficient matrix is totally unimodular.

(c) Let the decision variables be

$$x_{ij} = \begin{cases} 1 & \text{if worker } i \text{ does job } j \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 5, j = 1, ..., 5, \text{ and}$$
$$p_{ij} = \text{the percentage of job } j \text{ that worker } i \text{ does}, \ i = 1, ..., 5, j = 1, ..., 5.$$

The linear integer formulation is

$$\begin{aligned}
\min \quad & \sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij}p_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{5} p_{ij} = 1 \quad \forall j = 1, ..., 5 \\
& \sum_{i=1}^{5} x_{ij} \leq 2 \quad \forall j = 1, ..., 5 \\
& \sum_{j=1}^{5} x_{ij} \leq 2 \quad \forall i = 1, ..., 5 \\
& p_{ij} \leq x_{ij} \quad \forall i = 1, ..., 5, j = 1, ..., 5 \\
& x_{ij} \in \{0, 1\} \quad \forall i = 1, ..., 5, j = 1, ..., 5 \\
& p_{ij} \geq 0 \quad \forall i = 1, ..., 5, j = 1, ..., 5.
\end{aligned}$$

The first constraint guarantees that each job can be completely done by all workers. The second constraint guarantees that each job can be assigned to at most two workers. The third constraint guarantees that each worker can be assigned at most two kinds of jobs. The fourth constraint link the relationship between the two sets of variables.

4. The inventory-time graph is as shown below:

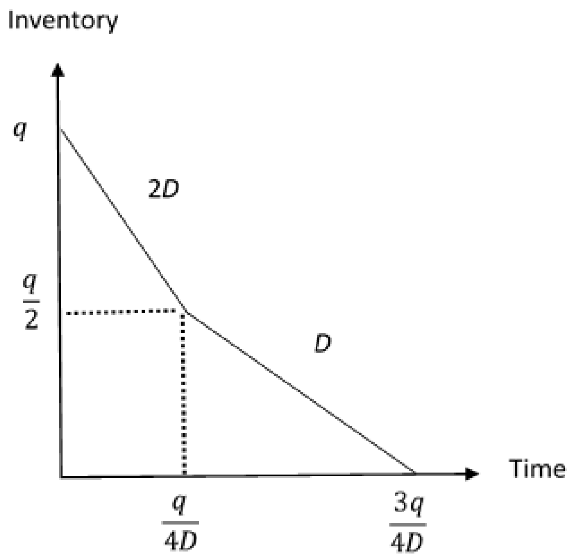

Figure 1: Inventory-time graph

The total annual cost C is the sum of annual ordering cost and annual holding cost.

$$\begin{aligned} C &= \frac{1}{\frac{3q}{4D}} \cdot K + \frac{\frac{5D}{16D}}{\frac{3q}{4D}} \cdot h \\ &= \frac{4KD}{3q} + \frac{5qh}{12} \end{aligned}$$

where the first one is annual ordering cost which is annual ordering times times ordering cost, and the second one is annual holding cost which is average inventory times holding cost. What we want to do is to minimize C, i.e.,

$$\min_q \ C = \frac{4KD}{3q} + \frac{5qh}{12}.$$

By FOC, we get optimal order quantity q^* is $\sqrt{\frac{16KD}{5h}}$.

5. (a) As $q^* = \sqrt{\frac{2KD}{h(1-\frac{D}{r})}} = \sqrt{\frac{(2)(50)(1000)}{1(1-\frac{1000}{1600})}} = 400\sqrt{\frac{5}{3}} = 516.397$, the EPQ is 516.397 units.

   (b) The overage cost $c_o = \$20$ and the underage cost $c_u = \$34$. Therefore, we have

$$1 - F(q^*) = \frac{c_o}{c_o + c_u} \quad \Rightarrow \quad \frac{200 - q^*}{200} = \frac{34}{54} \quad \Rightarrow \quad q^* = 206.6.$$

   the newsvender order quantity is 206.6 units.

6. (a) The formulation is

$$\begin{aligned} \max \quad & \sum_{i=1}^{2} (a_i - b_i q_i) q_i \\ \text{s.t.} \quad & q_1 \le q_2 \end{aligned}$$

(b) The Lagragian is

$$\mathcal{L}(q|\lambda) = (a_1 - b_1 q_1)q_1 + (a_2 - b_2 q_2)q_2 + \lambda(q_2 - q_1)$$

for some $\lambda \geq 0$. The FOC for the Lagrangian is

$$\frac{\partial}{\partial q_1}\mathcal{L}(q|\lambda) = a_1 - 2b_1 q_1 - \lambda = 0 \text{ and}$$
$$\frac{\partial}{\partial q_2}\mathcal{L}(q|\lambda) = a_2 - 2b_2 q_2 + \lambda = 0,$$

which imply

$$q_1 = \frac{a_1 - \lambda}{2b_1} \quad \text{and} \quad q_2 = \frac{a_2 + \lambda}{2b_2}.$$

When the constraint is binding,

$$q_2 - q_1 = 0 \quad \Leftrightarrow \quad \frac{a_2 + \lambda}{2b_2} - \frac{a_1 - \lambda}{2b_1} = 0 \quad \Leftrightarrow \quad \frac{a_2}{2b_2} - \frac{a_1}{2b_1} + \frac{\lambda}{2b_2} + \frac{\lambda}{2b_1} = 0$$

Since $\frac{\lambda}{2b_2} + \frac{\lambda}{2b_1} \geq 0$, we need $\frac{a_2}{2b_2} - \frac{a_1}{2b_1} \leq 0$ to satisfy the equality. Therefore, the condition for the constraint to be binding at an optimal solution is $\frac{a_2}{2b_2} \leq \frac{a_1}{2b_1}$.

(c) When the constraint is not binding, λ must be 0 and thus $q_i = \frac{a_i}{2b_i}$; otherwise, $\lambda \neq 0$ and $q_i = \frac{a_1+a_2}{2(b_1+b_2)}$ for $i = 1, 2$. Collectively, the optimal solution to this problem is

$$(q_1^*, q_2^*) = \begin{cases} \left(\frac{a_1}{2b_1}, \frac{a_2}{2b_2}\right) & \text{if } \frac{a_1}{2b_1} \leq \frac{a_2}{2b_2} \\ \left(\frac{a_1+a_2}{2(b_1+b_2)}, \frac{a_1+a_2}{2(b_1+b_2)}\right) & \text{otherwise} \end{cases}.$$

7. (a) The nonlinear program can be graphed as shown below

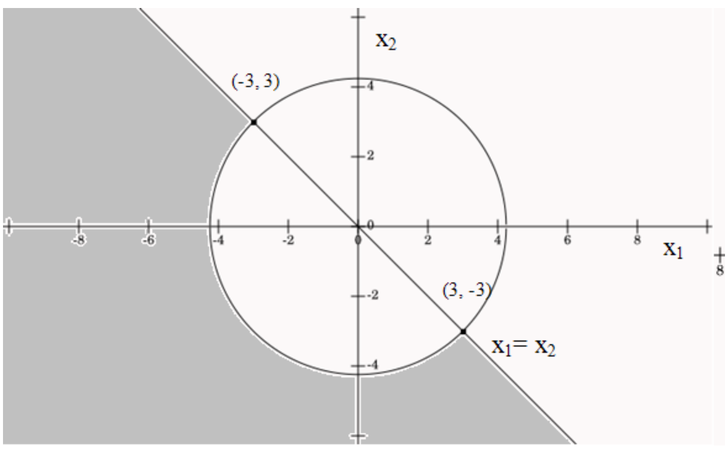

Figure 2: Graph for Problem 7

To maximize x_1 over this unbounded feasible region, the problem is unbounded.

(b) No. $(-3, 3)$ and $(3, -3)$ are in the feasible region, but a combination of these two points $\frac{1}{2}(-3, 3) + \frac{1}{2}(3, -3) = (0, 0)$ is not in the feasible region. Hence, it is not convex program.

(c) Yes. Let $f(x) = x_1, g_1(x) = x_1 + x_2$, and $g_2(x) = -x_1^2 - x_2^2$

i. Primal feasibility: $g_1(-3, 3) = -3 + 3 = 0 \leq 0$ and $g_2(-3, 3) = -9 - 9 \leq -18$.

ii. Dual feasibility: Given that $\nabla f(-3, 3) = (1, 0)$, $\nabla g_1(-3, 3) = (1, 1)$, and $\nabla g_2(-3, 3) = (6, -6)$, we need to find $\lambda \geq 0$ such that $\nabla f(-3, 3) = \lambda_1 \nabla g_1(-3, 3) + \lambda_2 \nabla g_2(-3, 3)$. It turns out that $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{1}{12}$ work.

iii. Complementary slackness: $\lambda_1[0 - g_1(-3,3)] = \frac{1}{2}[0-0] = 0$ and $\lambda_2[-18 - g_2(-3,3)] = \frac{1}{12}[-18+18] = 0$.

(d) No. Let $f(x) = x_1, g_1(x) = x_1 + x_2$, and $g_2(x) = x_1^2 + x_2^2$. For dual feasibility, given that $\nabla f(-3,3) = (1,0)$, $\nabla g_1(-3,3) = (1,1)$, and $\nabla g_2(-3,3) = (-6,6)$, we need to find $\lambda \geq 0$ such that $\nabla f(-3,3) = \lambda_1 \nabla g_1(-3,3) + \lambda_2 \nabla g_2(-3,3)$. As the unique solution has $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -\frac{1}{12}$, dual feasibility does not hold.

8. We define

$$x_t = \text{unit of product that produced in period } t, t = 1, ..., T, \text{ and}$$
$$y_t = \text{unit of product that sold in period } t, t = 1, ..., T$$

as our decision variables. The formulation is

$$\begin{aligned}
\max \quad & \sum_{t=1}^{T}(P_t y_t - C_t x_t) - S\sum_{i=1}^{T}\sum_{t=1}^{i}(D_t - y_t) - H\sum_{i=1}^{T}\sum_{t=1}^{i}(x_t - y_t) \\
\text{s.t.} \quad & \sum_{t=1}^{i} y_t \leq \sum_{t=1}^{i} x_t \quad \forall i = 1, ..., T \\
& \sum_{t=1}^{i} y_t \leq \sum_{t=1}^{i} D_t \quad \forall i = 1, ..., T \\
& \sum_{t=1}^{T} y_t \geq \sum_{t=1}^{T} D_t \\
& x_t \geq 0 \quad \forall t = 1, ..., T \\
& y_t \geq 0 \quad \forall t = 1, ..., T
\end{aligned}$$

The objective function maximizes the total profit. The first constraint ensures that the accumulated sales quantity is no more than the accumulated production quantity. The second constraint ensures that the accumulated sales quantity is no more than the accumulated demand quantity. The third constraint ensures that the total demand will be fulfilled eventually.