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Suggested Solution for Homework 8

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1. (a) Let $\lambda \in \mathbb{R}^2$ be the Lagrange multipliers for the two constraints. The Lagrangian is

$$\mathcal{L}(x|\lambda) = x_1 - x_2 + \lambda_1\Big(4 - x_1^2 - x_2^2\Big) + \lambda_2\Big(x_1^2 + (x_2+2)^2 - 4\Big)$$

and the FOC for the Lagrangian is

$$1 - 2\lambda_1 x_1 + 2\lambda_2 x_1 = 0 \quad \text{and} \quad -1 - 2\lambda_1 x_2 + 2\lambda_2(x_2+2) = 0.$$

Therefore, the KKT condition for this problem is the following: If a solution $\bar{x}$ is a local maximum, there exist $\lambda \in \mathbb{R}^2$ such that:

i. (Primal feasibility) $x_1^2 + x_2^2 \leq 4$ and $x_1^2 + (x_2+2)^2 \geq 4$.
ii. (Dual feasibility) $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $1 - 2\lambda_1 x_1 + 2\lambda_2 x_1 = 0$, and $-1 - 2\lambda_1 x_2 + 2\lambda_2(x_2+2) = 0$.
iii. (Complementary slackness) $\lambda_1(4 - x_1^2 - x_2^2) = 0$ and $\lambda_2(x_1^2 + (x_2+2)^2 - 4) = 0$.

(b) For the point to satisfy the KKT condition, first observe that primal feasibility and complementary slackness are both satisfied (as the two constraints are binding at $(\sqrt{3}, -1)$. For dual feasibility, we need

$$1 - 2\lambda_1\sqrt{3} + 2\lambda_2\sqrt{3} = 0 \quad \text{and} \quad -1 + 2\lambda_1 + 2\lambda_2 = 0,$$

whose solution is $\lambda_1 = \frac{3+\sqrt{3}}{12}$ and $\lambda_2 = \frac{3-\sqrt{3}}{12}$. As both multipliers are nonnegative, dual feasibility is also satisfied.

(c) Show that $(2, 0)$ violates the KKT condition.
First, it satisfies primal feasibility. Due to complementary slackness, we can see that λ_2 must be 0. Now, for dual feasibility, we need

$$1 - 2\lambda_1 = 0 \quad \text{and} \quad -1 = 0,$$

which is impossible. Therefore, $(2, 0)$ violates the KKT condition. Graphically, we can see that $(2, 0)$ is indeed not a local maximum.

Note. For an NLP, if a solution satisfies the KKT condition, it may or may not be a local optimum. However, if a solution violates the KKT condition, it cannot be a local optimum.

2. First of all, we can see that this is a convex program. Therefore, the KKT condition is necessary and sufficient and a local maximum is a global maximum. Let $\lambda \in \mathbb{R}$ be the Lagrange multiplier, the Lagrangian is

$$x_1 - x_2 + \lambda(4 - x_1^2 - x_2^2)$$

and the FOC for the Lagrangian is

$$1 - 2\lambda x_1 = 0 \quad \text{and} \quad -1 - 2\lambda x_2 = 0.$$

To satisfy these two equalities, we need $x_1 = -x_2$. Moreover, it is required that $\lambda > 0$. According to complementary slackness, we now know that $x_1^2 + x_2^2 = 4$. Therefore, to satisfy the KKT condition, we need to satisfy the two equalities

$$x_1 = -x_2 \quad \text{and} \quad x_1^2 + x_2^2 = 4.$$

The unique solution is $(\sqrt{2}, -\sqrt{2})$. As it satisfies the KKT condition for this convex program, it is an optimal solution.