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Suggested Solution for Homework 10

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1. (a) Replace t by $\frac{(A-Bw)^2}{4B}$, the objective function become

$$\begin{aligned}\pi_M^{**} &= \max_{w\geq 0, t\geq 0} \quad (w-C)(\frac{A-Bw}{2})+t \\ &= \max_{w\geq 0} \quad (w-C)(\frac{A-Bw}{2})+\frac{(A-Bw)^2}{4B} \\ &= \max_{w\geq 0} \quad (-\frac{B}{2}+\frac{B^2}{4B})w^2+... \\ &= \max_{w\geq 0} \quad (-\frac{B}{4})w^2+...\end{aligned}$$

Therefore, $\frac{d(\pi_M^{**})}{dw^2} = -\frac{B}{2} \leq 0$. Since the twice differentiation of π_M^{**} is small or equal to zero, the reduced problem is a convex program.

(b) By using FOC,

$$\begin{aligned}\frac{d(\pi_M^{**})}{dw} &= (-B+\frac{B}{2})w+\frac{A}{2}+\frac{BC}{2}-\frac{A}{2} \\ &= (-\frac{B}{2})w+\frac{BC}{2}.\end{aligned}$$

$w^* = C$, and the manufacturer earns $\frac{(A-BC)^2}{4B}$.

2. (a) The expected channel profit is

$$\begin{aligned}\pi_C(q) &= \int_0^q pxf(x)dx + \int_q^{100} pqf(x)dx - cq \\ &= \int_0^q pxf(x)dx + pq(1-F(q)) - cq.\end{aligned}$$

By using FOC, $F(q^*) = 1-\frac{c}{p} = 1-\frac{10}{50} = 0.8$. $q^* = 80, \pi_C(q^*) = \pi_C^{FB} = 1600$. Therefore, the efficient inventory level is 80 units ,and the efficient expected channel profit is \$1600.

(b) Under a wholesale contract, the retailer's expected payoff is

$$\pi_R^{(0)}(q) = \int_0^q pxf(x)dx + \int_q^{100} pqf(x)dx - wq$$

By using FOC, $q^* = 100-2w$. And the manufacturer's expected payoff is

$$\begin{aligned}\pi_M^{(0)}(q) &= (w-c)q \\ &= (w-10)(100-2w)\end{aligned}$$

By using FOC, $w^* = 30; q^* = 40; \pi_M^{(0)} = 800; \pi_R^{(0)} = 400$. The expected channel profit is $\pi_C^{(0)} = 1200 < \pi_C^{FB}$.

(c) Under a return contract with wholesale price \$30 and return credit \$5, the retailer's expected payoff is

$$\begin{aligned}\pi_R^{(1)}(q) &= \int_0^q pxf(x)dx + \int_q^{100} pqf(x)dx + \int_0^q 5(q-x)f(x)dx - wq \\ &= \int_0^q \frac{x}{2}dx + \int_q^{100} \frac{q}{2}dx + \int_0^q \frac{(q-x)}{20}dx - 30q \\ &= -\frac{9}{40}q^2 + 20q.\end{aligned}$$

By using FOC, $q^* = \frac{400}{9}$. The retailer's expected profit is $\pi_R^{(1)}(q^*) \approx \$444.44 > \pi_R^{(0)}$. And the manufacturer's expected payoff is

$$
\begin{aligned}
\pi_M^{(1)}(q) &= (w-c)q - \int_0^q 5(q-x)f(x)dx \\
&= 20\frac{400}{9} - \int_0^{\frac{400}{9}} 5(\frac{400}{9} - x)f(x)dx \\
&\approx 839.51 \\
&> \pi_M^{(0)}
\end{aligned}
$$

$\pi_C^{(1)} = 1283.95 > \pi_C^{(0)}$. Since $\pi_M^{(1)} > \pi_M^{(0)}$ and $\pi_R^{(1)} > \pi_R^{(0)}$, it is a win-win situation.

(d) Under a return contract with wholesale price \$30 and return credit \$10, the retailer's expected payoff is

$$
\begin{aligned}
\pi_R^{(2)}(q) &= \int_0^q pxf(x)dx + \int_q^{100} pqf(x)dx + \int_0^q 10(q-x)f(x)dx - wq \\
&= \int_0^q \frac{x}{2}dx + \int_q^{100} \frac{q}{2}dx + \int_0^q \frac{(q-x)}{10}dx - 30q \\
&= -\frac{1}{5}q^2 + 20q.
\end{aligned}
$$

By using FOC, $q^* = 50$. The retailer's expected profit is $\pi_R^{(2)}(q^*) = \$500 > \pi_R^{(1)}$. And the manufacturer's expected payoff is

$$
\begin{aligned}
\pi_M^{(2)}(q) &= (w-c)q - \int_0^q 10(q-x)f(x)dx \\
&= 1000 - \int_0^{50} 5(50-x)f(x)dx \\
&= 875 \\
&> \pi_M^{(1)}
\end{aligned}
$$

$\pi_C^{(2)} = 1375 > \pi_C^{(1)}$. Since $\pi_M^{(2)} > \pi_M^{(1)}$ and $\pi_R^{(2)} > \pi_R^{(1)}$, it is a win-win situation.

3. (a) When $\phi = 0$, the retailer gives nothing to the manufacturer; Therefore the revenue sharing contract becomes wholesale contract.

(b) The retailer's expected payoff is

$$
\pi_R(q) = (1-\phi)\left\{\int_0^q pxf(x)dx + \int_q^{\infty} pqf(x)dx\right\} - wq
$$

By using FOC, $F(q^*) = 1 - \frac{w}{p(1-\phi)}$. The retailer optimal order quantity $q^*(w,\phi) = F^{-1}(1 - \frac{w}{p(1-\phi)})$.

(c) When channel is coordinated, the channel optimal order quantity satisfies $F(q_C^*) = 1 - \frac{c}{p}$. And when $q^* = q_C^*, \phi = 1 - \frac{w}{c}$. Since $\phi \in [0,1], w$ should less or equal to c. To sum up, if the manaufacturer provides a revenue sharing contract with $(w,\phi) = (w, 1-\frac{w}{c})$ and $w \in [0,c]$ the channel will be coordinated.

4. Under ID:
We assume channel 1 has only one manufacturer (manufacturer 1), channel 2 has one manufacturer (manufacturer 2) and one retailer (retailer 2). First, for manufacturer 2 he should decide wholesale price w_2 to solve

$$
\pi_2^M = \max_{w_2} w_2 q_2.
$$

Then manufacturer 1 and retailer 2 should decide their retail price p_1, p_2 to solve

$$\pi_1^M = \max_{p_1} p_1 q_1$$
$$\pi_2^R = \max_{p_2} (p_2 - w_2) q_2.$$

Apply the backward induction, we solve second problem before first problem. By using FOC, we achieve

$$p_1^* = \frac{2+\theta+\theta w_2}{4-\theta^2}$$
$$p_2^* = \frac{2+\theta+2w_2}{4-\theta^2}$$

Then we solve manufacturer 2's problem and achieve

$$w_2^* = \frac{2+\theta}{2(2-\theta^2)}$$
$$p_1^* = \frac{4+\theta-2\theta^2}{(2-\theta)(2-\theta^2)}$$
$$p_2^* = \frac{3-\theta^2}{2(2-\theta)(2-\theta^2)}$$

Therefore $\pi_1^M = [\frac{4+\theta-2\theta^2}{2(2-\theta)(2-\theta^2)}]^2, \pi_2^M = \frac{2+\theta}{4(2-\theta)(2-\theta^2)}$.

5. Yes, DD may be an possible equilibrium industry structure. In II, two manufacturers play the Cournot game and consequently the equilibrium quantities are too high.If they change to DD, each channel has one additional layer and the quantity goes down since the wholesale price. Therefore we find that the system quantity is close to the optimal equilibrium quantity. .