Operations Research

Lab Session

TA: 陳嘉豪 (Jack Chen)
陳宗霆 (Tim Chen)
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Self-introduction

- 陳嘉豪(Jack Chen) : r02725018@ntu.edu.tw
- 陳宗霆(Tim Chen) : r02725052@ntu.edu.tw

Gauss-Jordan elimination (1)

(1) Use it to find a solution of an equation

Example:

$$x_1 + 2x_2 = 4$$

$$2x_1 + x_2 = 5$$

What is the solution of the equation?

Gauss-Jordan elimination (1)

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -3 & -3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$$\rightarrow (x_1, x_2) = (2, 1)$$

Gauss-Jordan elimination (1)

Practice:

$$
\begin{aligned}
x_1 + 2x_2 - 3x_3 &= 2 \\
x_1 \qquad\quad - x_3 &= 0 \\
x_1 - x_2 + 2x_3 &= 3
\end{aligned}
$$

What is the solution of the equation?

Gauss-Jordan elimination (1)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 2 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -2 & 2 & -2 \\ 0 & -3 & 5 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & -3 & 5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\rightarrow (x_1, x_2, x_3) = (2, 3, 2)$$

Gauss-Jordan elimination (2)

(2) Use it to find the inverse of a matrix

Example:

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

What is the inverse of matrix A?

Gauss-Jordan elimination (2)

$$\left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & -7 & -2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{-1}{7} \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{-3}{7} & \frac{5}{7} \\ 0 & 1 & \frac{2}{7} & \frac{-1}{7} \end{array}\right]$$

$$\rightarrow A^{-1} = \frac{1}{7}\begin{bmatrix} -3 & 5 \\ 2 & -1 \end{bmatrix}$$

Gauss-Jordan elimination (2)

Practice:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 1 & -2 \\ 3 & 1 & -1 \end{bmatrix}$$

What is the inverse of matrix A?

Gauss-Jordan elimination (2)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ 0 & 1 & 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \end{array}\right] \rightarrow \quad A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -2 & -4 & 6 \\ 1 & -1 & 1 \end{bmatrix}$$

Linearly independent / dependent

- $A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \end{bmatrix}$
- Column vectors of $A = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix}\}$
- Row vectors of $A = \{[1 \quad 2 \quad 5], [0 \quad 1 \quad 2]\}$

- Definition: A collection of vectors $a^1,\ldots,a^n$ is linearly independent if $\sum_{j=1}^{n} c_j a^j = 0$ imply that $c_1 = c_2 = \ldots = c_n = 0$

- $1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  => Column vectors of A are linearly dependent
- $c_1[1 \quad 2 \quad 5] + c_2[0 \quad 1 \quad 2] = 0$ only when $c_1 = c_2 = 0$

  => Row vectors of A are linearly independent

Linearly independent / dependent

- When row vectors of a matrix are linearly dependent
- After Gauss-Jordan elimination

  there must exist at least a row that all element are 0.

Example:

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{No inverse!!!}$$

Rank of a matrix

- **Column rank** of matrix A is the maximal number of linearly independent column of A.
- **Row rank** of matrix A is the maximal number of linearly independent row of A.
- Since the column rank and row rank are always equal, they are simply called the rank of A.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 2 & -1 \end{bmatrix} \rightarrow \text{Rank} = 2, \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 4 \end{bmatrix} \rightarrow \text{Rank} = 1$$

Rank of a matrix

- $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & 1 & 5 \end{bmatrix}$, rank($A$)= ? A^{-1}?
- $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$
- Rank(A) $=$ 2, no inverse!

DFSI principle

- Step 1: Define the decision variables (and the notations you use for parameters).
- Step 2: Formulate the problem as a mathematical model by writing down the objective function and constraints.
- Step 3: Solve the model by finding the values for all decision variables in an optimal solution.
- Step 4: Interpret the optimal solution by indicating "what to do".

DFSI principle

Scenario:

- You are in a market, and you have 5 dollars.
- There are several drinks (Coke, Pepsi, Orange juice).
- Each drink gives you different happiness level.
- Which drinks should you buy?

DFSI principle

Step 1: Define the problem and collect relevant data

Goal: To maximize your happiness level.

Data:

Name	Price	Happiness level
Coke	2 dollars	3
Pepsi	3 dollars	6
Orange juice	2.5 dollars	4

DFSI principle

Step 2: Formulating the problem

Parameters: 5 dollars, 3 drinks….

Decision variables: For each drink, we decide whether to buy.
Let x_i be the amount of the drink we buy, i =1,2,3.

Objected function: $3x_1 + 6x_2 + 4x_3$

Constraint: $2x_1 + 3x_2 + 2.5x_3 \leq 5$

Our model:

$$\begin{aligned} \max \quad & 3x_1 + 6x_2 + 4x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 2.5x_3 \leq 5 \\ & x_1, x_2, x_3 \in \{0, 1\} \end{aligned}$$

DFSI principle

Step 3: Solving the model

(1, 1, 0) will be the solution.

Objected value is 9

Step 4: Interpret

To get maximum happiness, we should buy a Coke and a Pepsi.

Finally we get 9 happiness level.

Thank you ☺