IM 2010: Operations Research, Spring 2014

Nonlinear Programming (Part 2)

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Road map

- **Multi-variate convex analysis**.
- Solving constrained NLPs.
- Applications.

Convex analysis

- We have learned how to solve single-variate NLPs.
  - An optimal solution either satisfies the **FOC** or is a boundary point.
  - If the NLP is a **CP**, a feasible point satisfying the FOC is optimal.
- The above facts actually apply to **multi-variate NLPs**.
- We need to be able to determine whether a multi-variate function is convex, concave, or neither.
- We will still focus on **twice differentiable** functions.
  - Let's extend the notion of derivatives first.

Partial derivatives

- For a function $f : \mathbb{R}^n \to \mathbb{R}$, its ith **partial derivative** is $\frac{\partial f(x)}{\partial x_i}$.
  - E.g., the partial derivatives for

$$f(x_1, x_2, x_3) = x_1^2 + x_2 x_3 + x_3^3$$

are

$$\frac{\partial f(x)}{\partial x_1} = 2x_1, \frac{\partial f(x)}{\partial x_2} = x_3 \text{ and } \frac{\partial f(x)}{\partial x_3} = x_2 + 3x_3^2.$$

- It also has **second-order partial derivatives**:
  - For the same f, we have

$$\frac{\partial^2 f(x)}{\partial x_1^2} = 2, \frac{\partial^2 f(x)}{\partial x_2^2} = 0, \frac{\partial^2 f(x)}{\partial x_3^2} = 6x_3,$$

$$\frac{\partial^2 f(x)}{\partial x_1 x_2} = \frac{\partial^2 f(x)}{\partial x_2 x_1} = 0, \frac{\partial^2 f(x)}{\partial x_1 x_3} = \frac{\partial^2 f(x)}{\partial x_3 x_1} = 0, \frac{\partial^2 f(x)}{\partial x_2 x_3} = \frac{\partial^2 f(x)}{\partial x_3 x_2} = 1.$$

Symmetry of second-order derivatives

- For a second-order derivatives, we have the following fact:

Proposition 1

For a twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, if its second-order derivatives are all continuous, then

$$\frac{\partial^2 f(x)}{\partial x_i x_j} = \frac{\partial^2 f(x)}{\partial x_j x_i}$$

for all $i = 1, ..., n$, $j = 1, ..., n$.

- For all functions we will see in this course, the above property holds.

Multi-variate convex functions

- For $f : \mathbb{R} \to \mathbb{R}$, f is convex if and only if $f''(x) \geq 0$ for all x.
- For $f : \mathbb{R}^n \to \mathbb{R}$, is it true that f is convex if and only if $\frac{\partial^2 f(x)}{\partial x_i^2} \geq 0$ for all x_i, $i = 1, ..., n$?
- Consider $f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + x_1 + x_2$. Is it convex at $(0, 0)$?
  - We have

$$\frac{\partial f(0,0)}{\partial x_1} = (2x_1 + 4x_2 + 1)\bigg|_{(x_1,x_2)=(0,0)} = 1 \quad \text{and} \quad \frac{\partial^2 f(0,0)}{\partial x_1^2} = 2 > 0.$$

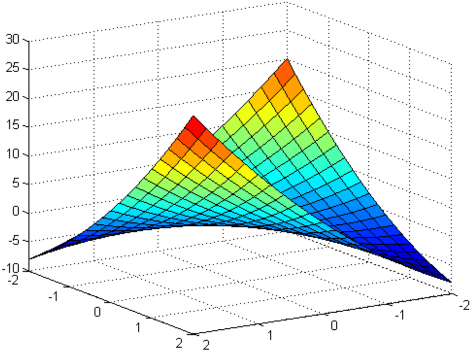
  - We also have

$$\frac{\partial f(0,0)}{\partial x_2} = (2x_2 + 4x_1 + 1)\bigg|_{(x_1,x_2)=(0,0)} = 1 \quad \text{and} \quad \frac{\partial^2 f(0,0)}{\partial x_1^2} = 2 > 0$$

  - Is f convex at $(0, 0)$?

Multi-variate convex functions

- This is necessary but **insufficient**!
- $\frac{\partial^2}{\partial x_1^2} f(0,0) \geq 0$ and $\frac{\partial^2}{\partial x_2^2} f(0,0) \geq 0$ only imply that f is convex **along the two axes**!
  - Along $(1, -1)$, e.g., f is not convex.
- We need to test whether f is convex **in all directions**.



$$f(x_1, x_2) = x_1^2 + 4x_1x_2 + x_2^2 + x_1 + x_2.$$

Gradients and Hessians

- For a function $f : \mathbb{R}^n \to \mathbb{R}$, collecting its first- and second-order partial derivatives generates its **gradient** and **Hessian**:

Definition 1 (Gradients and Hessians)

For a multi-variate twice differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, its gradient and Hessian are

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \vdots \\ \frac{\partial f(x)}{\partial x_n} \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \ddots & \\ \vdots & & \frac{\partial^2 f(x)}{\partial x_n^2} \end{bmatrix}.$$

- In this course, all Hessians are **symmetric**.

Example

- For $f(x_1, x_2, x_3) = x_1^2 + x_2 x_3 + x_3^3$, the gradient is

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f(x)}{\partial x_1} \\ \frac{\partial f(x)}{\partial x_2} \\ \frac{\partial f(x)}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_3 \\ x_2 + 3x_3^2 \end{bmatrix}.$$

- The Hessian is

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_2^2} & \frac{\partial^2 f(x)}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f(x)}{\partial x_3 \partial x_1} & \frac{\partial^2 f(x)}{\partial x_3 \partial x_2} & \frac{\partial^2 f(x)}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 6x_3 \end{bmatrix}.$$

- What are $\nabla f(3,2,1)$ and $\nabla^2 f(3,2,1)$?

Convexity of twice differentiable functions

- Recall the following theorem for single-variate functions:

Proposition 2

For a single-variate twice differentiable function $f(x)$:

- *f is convex in $[a, b]$ if $f''(x) \geq 0$ for all $x \in [a, b]$.*
- *$\bar{x}$ is an interior local min only if $f'(\bar{x}) = 0$.*
- *If f is convex, x^* is a global min if and only if $f'(x^*) = 0$.*

- We have an analogous theorem for multi-variate functions:

Proposition 3

For a multi-variate twice differentiable function $f(x)$:

- *f is convex in F if $\nabla^2 f(x)$ is positive semi-definite for all $x \in F$.*
- *$\bar{x}$ is an interior local min only if $\nabla f(x) = 0$.*
- *If f is convex, x^* is a global min if and only if $\nabla f(x^*) = 0$.*

  - What is **positive semi-definiteness** (PSD)?

Positive semi-definite matrices

- Positive semi-definite Hessians in $\mathbb{R}^n$ are **generalizations** of nonnegative second-order derivatives in $\mathbb{R}$.

Definition 2 (Positive semi-definite matrices)

A symmetric matrix A is positive semi-definite if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$.

- Example 1: For $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, we have

$$x^T Ax = 2x_1^2 + 2x_1x_2 + 2x_2^2 = (x_1 + x_2)^2 + x_1^2 + x_2^2 \geq 0 \quad \forall x \in \mathbb{R}^2.$$

- Example 2: For $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, we have $x^T Ax = x_1^2 + 4x_1x_2 + x_2^2$, which is negative when $x_1 = 1$ and $x_2 = -1$.

Positive semi-definite matrices

- Given a function f, when is its Hessian $\nabla^2 f$ PSD?

Proposition 4

For a symmetric matrix A, the following statements are equivalent:

- *A is positive semi-definite.*
- *A's eigenvalues are all nonnegative.*
- *A's leading principal minors are all nonnegative.*

  - A's eigenvalues λ and eigenvectors x satisfy $Ax = \lambda x$.
  - A's kth leading principal minors is the determinant of the upper-left k by k submatrix.
- Given a function f, we will:
  - Find its Hessian.
  - Find its eigenvalues or leading principal minors.
  - Determine over what region the Hessian is PSD.
  - Over that region, the function is convex.

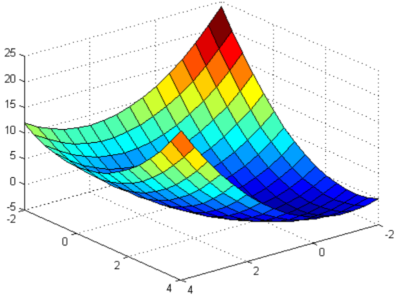
An example

- Consider the NLP

$$\min_{x \in \mathbb{R}^2} f(x_1, x_2),$$

where

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2 - 2x_1 - 4x_2.$$



- Its gradient and Hessian are

$$\nabla f(x_1, x_2) = \begin{bmatrix} 2x_1 + x_2 - 2 \\ x_1 + 2x_2 - 4 \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x_1, x_2) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

An example

$$f(x_1, x_2) = x_1^2 + x_2^2 + x_1 x_2 - 2x_1 - 4x_2.$$

- To find the eigenvalues of $\nabla^2 f(x_1, x_2)$, recall that

$$Ax = \lambda x \quad \Leftrightarrow \quad (A - \lambda I)x = 0 \quad \Leftrightarrow \quad \det(A - \lambda I) = 0.$$

- For our $\nabla^2 f(x_1, x_2)$, we have

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad 3 - 4\lambda + \lambda^2 = 0 \quad \Leftrightarrow \quad \lambda = 1 \text{ or } 3.$$

- Or by leading principal minors:

$$\begin{vmatrix} 2 \end{vmatrix} = 2 \quad \text{and} \quad \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

- So $\nabla^2 f(x_1, x_2)$ is PSD and thus $\min_{x \in \mathbb{R}^2} f(x_1, x_2)$ is a CP. The FOC requires $2x_1^* + x_2^* - 2 = 0$ and $x_1^* + 2x_2^* - 4 = 0$, i.e., $(x_1^*, x_2^*) = (0, 2)$.

Another example

- Consider $f(x_1, x_2) = x_1^3 + 4x_1x_2 + x_2^2 + x_1 + x_2$. When is it convex?
- Its Hessian is
$$\begin{bmatrix} 6x_1 & 4 \\ 4 & 1 \end{bmatrix}.$$
- When is the Hessian positive semi-definite?
  - We need the first leading principal minor $6x_1 \geq 0$.
  - We need the second leading principal minor $6x_1 - 16 \geq 0$.
- Therefore, the function is convex if and only if $x_1 \geq \frac{8}{3}$.

Road map

- Multi-variate convex analysis.
- **Solving constrained NLPs**.
- Applications.

Solving constrained NLPs

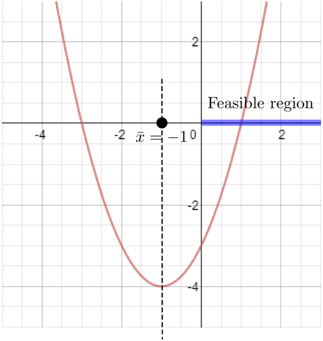
- For **unconstrained NLPs**, we have enough tools:
  - We may determine whether the objective function is convex.
  - We may use the FOC to find all local minima.
- How about **constrained NLPs**?
- We may always try the following strategy:
  - Ignore all the constraints.
  - Find a global minimum.
  - If it is feasible, it is optimal.
- It an unconstrained global minimum is infeasible, what should we do?

Solving single-variate constrained NLPs

- Let's solve

$$\min_{x \geq 0} \; f(x) = x^2 + 2x - 3.$$

- We have $f'(x) = 2x + 2$ and $f''(x) = 2$.
- f is convex and the solution satisfying the FOC is $\bar{x} = -1$. However, it is infeasible!
- For a single-variate NLP, the feasible solution that is **closest** to the FOC-solution is optimal.



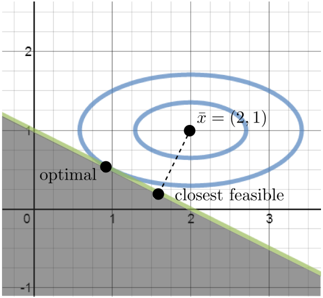
$$f(x) = x^2 + 2x - 3.$$

Solving multi-variate constrained NLPs

- Let's solve

$$\min_{x \in \mathbb{R}^2} \quad f(x) = (x_1 - 2)^2 + 4(x_2 - 1)^2$$
$$\text{s.t.} \quad x_1 + 2x_2 \leq 2.$$

- For this CP, the FOC-solution $\bar{x} = (2, 1)$ is infeasible.
- The closest feasible point is **not** optimal!
- We need a way to deal with constraints.



$$f(x) = x^2 + 2x - 3.$$

Relaxation with rewards

- Recall our strategy: First ignore all constraints, and then ...
- Ignoring all constraints is "too much"!
  - An infeasible solution should be bad.
  - But this cannot be revealed in the relaxed NLP.
  - While we allow one to violate constraints, we **encourage** feasibility.
- Consider an original NLP

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq b_i \quad \forall i = 1, ..., m. \end{aligned}$$

- How to allow one to violate constraints but encourage feasibility?
  - For constraint i, let's associate a unit **reward** $\lambda_i \geq 0$ to it.
  - If a solution $\bar{x}$ satisfies constraint i (so $b_i - g_i(\bar{x}) \geq 0$), "reward" the solution by $\lambda_i[b_i - g_i(\bar{x})]$. Let's add this into the relaxed NLP.

Lagrangian relaxation

- For an original NLP

$$z^* = \max_{x \in \mathbb{R}^n} \left\{ f(x) \middle| g_i(x) \leq b_i \ \forall i = 1, ..., m \right\}, \quad (1)$$

we relax the constraints and add **rewards for feasibility** into the objective function:

$$z^L(\lambda) = \max_{x \in \mathbb{R}^n} \ f(x) + \sum_{i=1}^{m} \lambda_i \Big[b_i - g_i(x) \Big]. \quad (2)$$

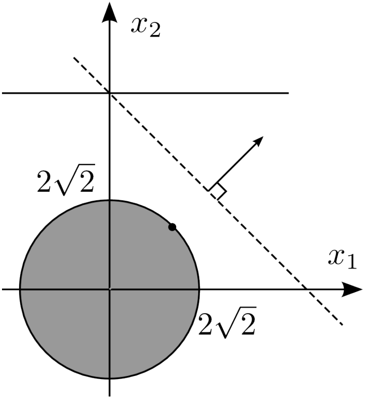
  - Let's assume that λ_is are given for a while.
  - To help solve the NLP, we should have $\lambda_i \geq 0$. This **rewards feasibility** and **penalize infeasibility**.
- $\mathcal{L}(x|\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i [b_i - g_i(x)]$ is the **Lagrangian** given λ.
- λ_is are the **Lagrange multipliers**.

An example

- Consider the following example

$$\begin{aligned} z^* = \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 8 \\ & x_2 \leq 6. \end{aligned}$$

- For this original NLP, the optimal solution is $x^* = (2, 2)$. $z^* = 4$.
- What are the Lagrangian and Lagrangian relaxation?



An example

- The original NLP is $z^* = \max_{x \in \mathbb{R}^2} \left\{ x_1 + x_2 \middle| x_1^2 + x_2^2 \leq 8, x_2 \leq 6 \right\}$.
- Given Lagrange multipliers $\lambda = (\lambda_1, \lambda_2) \geq 0$, the Lagrangian is

$$\mathcal{L}(x|\lambda) = x_1 + x_2 + \lambda_1(8 - x_1^2 - x_2^2) + \lambda_2(6 - x_2).$$

- The Lagrangian relaxation is

$$z^L(\lambda) = \max_{x \in \mathbb{R}^2} \mathcal{L}(x|\lambda).$$

- Some Lagrange multipliers:
  - $z^L(0,1) = \max_{x \in \mathbb{R}^2} \; x_1 + 6 = \infty$.
  - $z^L(1,2) = \max_{x \in \mathbb{R}^2} \; -x_1^2 + x_1 - x_2^2 - x_2 + 20 = 20.5$.
  - $z^L(1,0) = \max_{x \in \mathbb{R}^2} \; -x_1^2 + x_1 - x_2^2 - x_2 + 8 = 8.5$.
- All the $z^L(\lambda)$ above is greater than $z^* = 4$! Will this always be true?

Lagrangian relaxation provides a bound

- The Lagrangian relaxation provides a **bound** for the original NLP.

Proposition 5

For the two NLPs defined in (1) and (2), $z^L(\lambda) \geq z^*$ for all $\lambda \geq 0$.

Proof. We have

$$\begin{aligned} z^* &= \max_{x \in \mathbb{R}^n} \left\{ f(x) \middle| g_i(x) \leq b_i \ \forall i = 1, ..., m \right\} \\ &\leq \max_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda_i [b_i - g_i(x)] \middle| g_i(x) \leq b_i \ \forall i = 1, ..., m \right\} \\ &\leq \max_{x \in \mathbb{R}^n} \left\{ f(x) + \sum_{i=1}^m \lambda_i [b_i - g_i(x)] \right\} = z^L(\lambda), \end{aligned}$$

where the first inequality relies on $\lambda \geq 0$. □

Lagrangian duality

- Given a constrained original NLP, solving its Lagrangian relaxation gives us some information.
- A similar situation happened to LP!
  - Any feasible dual solution gives a bound to the primal LP.
  - We look for an dual optimal solution that gives a tight bound.
- Given that $z^L(\lambda) \geq z^*$ for all $\lambda \geq 0$, it is natural to define

$$\min_{\lambda \geq 0} \ z^L(\lambda)$$

  as the **Lagrangian dual program**.
  - Lagrange multipliers are **dual variables** in NLP.
  - LP duality is a special case of Lagrangian duality: The Lagrangian relaxation of an LP is the dual LP.
- Lagrangian duality possesses several properties (beyond the scope).
  - Just intuitively treat λ_i as the dual variable for constraint i.

The KKT condition

- Now we present the most useful optimality condition for general NLPs:

Proposition 6 (KKT condition)

For a "regular" NLP

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & f(x) \\ s.t. \quad & g_i(x) \leq b_i \quad \forall i = 1, ..., m. \end{aligned}$$

if $\bar{x}$ is a local max, then there exists $\lambda \in \mathbb{R}^m$ such that

- $g_i(\bar{x}) \leq b_i$ *for all* $i = 1, ..., m$,
- $\lambda \geq 0$ *and* $\nabla f(\bar{x}) = \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x})$, *and*
- $\lambda_i [b_i - g_i(\bar{x})] = 0$ *for all* $i = 1, ..., m$.

- All NLPs in this course (and most in the world) are "regular".
- The condition is necessary for general NLPs but also sufficient for CPs.

The KKT condition

- There are three conditions for $\bar{x}$ to be a local maximum.
- **Primal feasibility**: $g_i(\bar{x}) \leq b_i$ for all $i = 1, ..., m$.
  - It must be feasible.
- **Dual feasibility**: $\lambda \geq 0$ and $\nabla f(\bar{x}) = \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x})$.
  - The equality is the **FOC for the Lagrangian** $\mathcal{L}(\bar{x}|\lambda)$:

$$\nabla \left\{ f(x) + \sum_{i=1}^{m} \lambda_i [b_i - g_i(x)] \right\} = 0 \quad \Leftrightarrow \quad \nabla f(\bar{x}) - \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0.$$

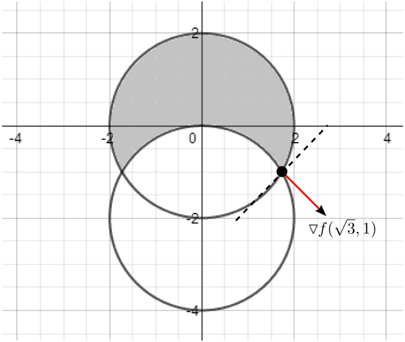
- **Complementary slackness**: $\lambda_i [b_i - g_i(\bar{x})] = 0$ for all $i = 1, ..., m$.
  - Dual variable $\times$ primal slack $= 0$.
  - If a constraint is **nonbinding**, the Lagrange multiplier is 0.
- Let's visualize the KKT condition.

Visualizing the KKT condition

- Consider

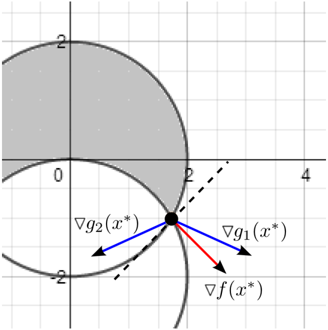
$$\begin{aligned} \max \quad & x_1 - x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 4 \\ & -x_1^2 - (x_2+2)^2 \leq -4. \end{aligned}$$

- Graphically, $x^* = (\sqrt{3}, 1)$ is optimal.
- What happens to ∇f, ∇g_1, and ∇g_2 at x^*?



Visualizing the KKT condition

$$\begin{aligned} \max \quad & f(x) = x_1 - x_2 \\ \text{s.t.} \quad & g_1(x) = x_1^2 + x_2^2 \leq 4 \\ & g_2(x) = -x_1^2 - (x_2+2)^2 \leq -4. \end{aligned}$$



- We have $\nabla f(x) = (1, -1)$, $\nabla g_1(x) = (2x_1, 2x_2)$, and $\nabla g_2(x) = (-2x_1, -2(x_2+2))$,
- Therefore, $\nabla f(x^*) = (1, -1)$, $\nabla g_1(x^*) = (2\sqrt{3}, -2)$, and $\nabla g_2(x^*) = (-2\sqrt{3}, -2)$.
- The existence of $\lambda \geq 0$ such that $\nabla f(x^*) = \lambda_1 \nabla g_1(x^*) + \lambda_2 \nabla g_2(x^*)$ simply means that ∇f is "**in between**" ∇g_1 and ∇g_2 at x^*.
  - Otherwise there is a feasible improving direction.
  - Complementary slackness $\lambda_i[b_i - g_i(x^*)]$ says that only constraints binding at x^* matter.

Applying the KKT condition

$$\begin{aligned}\max \quad & f(x) = x_1 - x_2 \\ \text{s.t.} \quad & g_1(x) = x_1^2 + x_2^2 \leq 4 \\ & g_2(x) = -x_1^2 - (x_2+2)^2 \leq -4.\end{aligned}$$

- The Lagrangian is

$$\mathcal{L}(x|\lambda) = x_1 - x_2 + \lambda_1(4 - x_1^2 - x_2^2) + \lambda_2(-4 + x_1^2 + (x_2+2)^2).$$

- $\frac{\partial \mathcal{L}(x|\lambda)}{\partial x_1} = 1 - 2(\lambda_1 - \lambda_2)x_1$ and $\frac{\partial \mathcal{L}(x|\lambda)}{\partial x_2} = -1 - 2(\lambda_1 - \lambda_2)x_2 + 4\lambda_2$.
- A solution $\bar{x}$ is a local maximum only if there exists λ such that

$$\begin{aligned}& x_1^2 + x_2^2 \leq 4, -x_1^2 - (x_2+2)^2 \leq -4 \\ & \lambda_1 \geq 0, \lambda_2 \geq 0 \\ & 1 - 2(\lambda_1 - \lambda_2)x_1 = 0, -1 - 2(\lambda_1 - \lambda_2)x_2 + 4\lambda_2 = 0 \\ & \lambda_1(4 - x_1^2 - x_2^2) = 0, \lambda_2(-4 + x_1^2 + (x_2+2)^2) = 0.\end{aligned}$$

The KKT condition for analysis

- In general, if there are n variables and m constraints.
  - There are n primal variables (x) and m dual variables (λ).
  - There are n equalities for dual feasibility.
  - There are m equalities for complementary slackness.
- As those equalities are nonlinear, there may be multiple solutions satisfying those equalities.
  - Those inequalities are then used to eliminate some solutions.
- If we have all local maxima, we compare them for a global maximum.
  - Nonlinear equations are hard to solve (even numerically).
  - Too time consuming in general.
- Nevertheless, we will see that the KKT condition is useful for analyzing many problems in business and economics.

Road map

- Multi-variate convex analysis.
- Solving constrained NLPs.
- **Applications**.

Multi-product EOQ problem

- Recall that we have solved the EOQ problem

$$\min_{q \geq 0} \frac{hq}{2} + \frac{KD}{q},$$

where h is the unit holding cost per year, K is the ordering cost per order, and D is the annual demand. The EOQ is $q^* = \sqrt{\frac{2KD}{h}}$.

- What if we procure two products? We solve

$$\min_{q_1 \geq 0, q_2 \geq 0} \frac{h_1 q_1}{2} + \frac{K_1 D_1}{q_1} + \frac{h_2 q_2}{2} + \frac{K_2 D_2}{q_2}.$$

The problem is separable; the optimal quantities are the two EOQs.

Multi-product EOQ problem

- What if we have only a limited space for these two products?
- We solve

$$\begin{array}{rl}\min\limits_{q_1 \geq 0, q_2 \geq 0} & \dfrac{h_1 q_1}{2} + \dfrac{K_1 D_1}{q_1} + \dfrac{h_2 q_2}{2} + \dfrac{K_2 D_2}{q_2} \\ \text{s.t.} & v_1 q_1 + v_2 q_2 \leq W,\end{array}$$

  where W is the total space and v_i is the volume of product i.
- Assumptions:
  - We assume that products can be "in any shape".
  - This constraint can also be modeling budgets or something else.
  - We do not try to "synchronize" the procurement processes (so we assume the orders for the two products may arrive at the same time).
- How to solve this problem?
- To simplify the derivation, assume that $v_1 = v_2 = 1$ and $h_1 = h_2 = h$.

Convexity of the problem

- Our (simplified) two-product EOQ problem

$$\begin{array}{rl} \min\limits_{q_1 \geq 0, q_2 \geq 0} & \frac{hq_1}{2} + \frac{K_1 D_1}{q_1} + \frac{hq_2}{2} + \frac{K_2 D_2}{q_2} \\ \text{s.t.} & q_1 + q_2 \leq W, \end{array}$$

  is a CP:
  - The objective function is convex; the Hessian matrix

$$\begin{bmatrix} \frac{K_1 D_1}{q_1^2} & 0 \\ 0 & \frac{K_2 D_2}{q_2^2} \end{bmatrix}$$

    is positive semi-definite.
  - The feasible region is convex.
  - A local minimum is a global minimum.

The FOC for the Lagrangian

- The Lagrangian is

$$\mathcal{L}(q|\lambda) = \frac{hq_1}{2} + \frac{K_1 D_1}{q_1} + \frac{hq_2}{2} + \frac{K_2 D_2}{q_2} + \lambda(W - q_1 - q_2).$$

- The FOC for the Lagrangian is

$$\frac{\partial}{\partial q_1}\mathcal{L}(q|\lambda) = \frac{h}{2} - \frac{K_1 D_1}{q_1^2} - \lambda = 0 \text{ and}$$
$$\frac{\partial}{\partial q_2}\mathcal{L}(q|\lambda) = \frac{h}{2} - \frac{K_2 D_2}{q_2^2} - \lambda = 0.$$

  Note that this must be satisfied by **any optimal solution**!

- Therefore, we have

$$\frac{K_1 D_1}{q_1^2} = \frac{K_2 D_2}{q_2^2} \quad \Leftrightarrow \quad \frac{q_1}{q_2} = \sqrt{\frac{K_1 D_1}{K_2 D_2}}.$$

Solving the multi-product EOQ problem

- Now we are ready to solve our two-product EOQ problem

$$\min_{q_1 \geq 0, q_2 \geq 0} \left\{ \frac{hq_1}{2} + \frac{K_1 D_1}{q_1} + \frac{hq_2}{2} + \frac{K_2 D_2}{q_2} \middle| q_1 + q_2 \leq W \right\}.$$

- If the unconstrained optimal solution $(\bar{q}_1, \bar{q}_2) = \left(\sqrt{\frac{2K_1 D_1}{h}}, \sqrt{\frac{2K_2 D_2}{h}} \right)$ satisfies $\bar{q}_1 + \bar{q}_2 \leq W$, it is optimal.
- Otherwise, the capacity constraint must be binding. The solution to the two equalities

$$q_1 + q_2 = W \quad \text{and} \quad \frac{q_1}{q_2} = \sqrt{\frac{K_1 D_1}{K_2 D_2}}$$

is optimal; i.e., $(\tilde{q}_1, \tilde{q}_2) = \left(\frac{W}{1 + \sqrt{\frac{K_2 D_2}{K_1 D_1}}}, \frac{W}{1 + \sqrt{\frac{K_1 D_1}{K_2 D_2}}} \right)$ is optimal.

Solving the multi-product EOQ problem

- Collectively, the optimal solution is

$$(q_1^*, q_2^*) = \begin{cases} \left(\sqrt{\frac{2K_1D_1}{h}}, \sqrt{\frac{2K_2D_2}{h}}\right) & \text{if } \sqrt{\frac{2K_1D_1}{h}} + \sqrt{\frac{2K_2D_2}{h}} \leq W \\ \left(\frac{W}{1+\sqrt{\frac{K_2D_2}{K_1D_1}}}, \frac{W}{1+\sqrt{\frac{K_1D_1}{K_2D_2}}}\right) & \text{otherwise.} \end{cases}$$