IM 2010: Operations Research, Spring 2014

Game Theory (Part 2): Dynamic Games

Ling-Chieh Kung

Department of Information Management
National Taiwan University

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Road map

- **Basic ideas**.
- Pricing in a supply chain.
- Indirect newsvendors.

Dynamic BoS

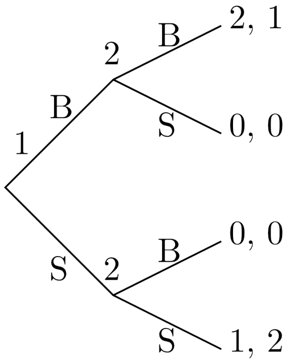
- Recall the game "Bach or Stravinsky":

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- What if the two players act **sequentially** instead of simultaneously?
  - What will they do in equilibrium?
  - How do their payoffs change?
  - Is it better to be the **leader** or the **follower**?

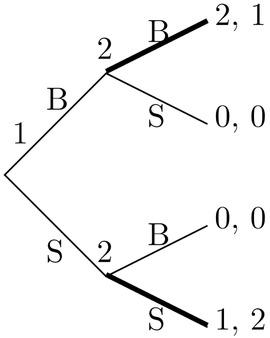
Dynamic BoS

- Suppose player 1 **moves** first.
- Instead of a game matrix, the game can now be described by a **game tree**.
  - At each internal node, the label shows who is taking an action.
  - At each link, the label shows an action.
  - At each leaf, the numbers show the payoffs.
- The games is played from the root to leaves.



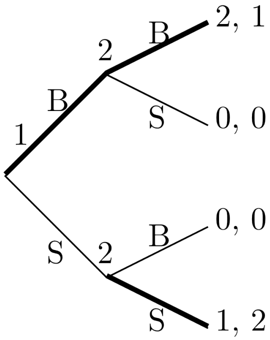
Dynamic BoS: Player 2's strategy

- How should player 1 move?
  - She needs to first **predict** how player 2 will response.
- She first treats herself as player 2:
  - If B has been chosen, choose B.
  - If S has been chosen, choose S.
- This is exactly player 2's **best response** to player 1's action.
  - It is also player 2's optimal strategy.
- We use thick lines to mark player 2's optimal strategy.



Dynamic BoS: Player 1's strategy

- How should player 1 move?
  - She knows how player 2 reacts.
  - Based on that, she chooses her action.
- Player 1 thinks:
  - If I choose B, I will end up with 2.
  - If I choose S, I will end up with 1.
- So player 1 will choose B.
- We also use a thick line to mark player 1's optimal strategy.
- A thick line that connects the root and a leave marks an **equilibrium outcome**.
  - In equilibrium, they play (B, B).



Dynamic BoS vs. static BoS

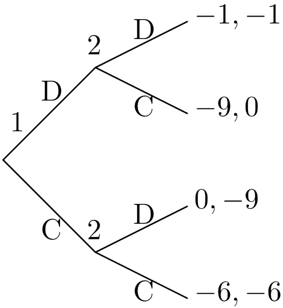
- Regarding predicting their behaviors:
  - In the static case, we cannot perfectly predict what they will do.
  - But in the dynamic case, we can!
  - Their **equilibrium behaviors** change.
- Questions:
  - Do the equilibrium behaviors always change when we switch from a static game to a dynamic game?
  - What if player 2 is the leader and player 1 is the follower?

Dynamic prisoners' dilemma

- Recall the game "prisoners' dilemma":

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

  The equilibrium outcome is (Confession, Confession).
- What if they move sequentially?
- In equilibrium, they will **both confess**.
  - The outcome **does not change**!
  - Even if they have agreed to both deny, player 1 has denied, and player 2 has observed it, player 2 will still confess.
  - Player 1's promise is useless.

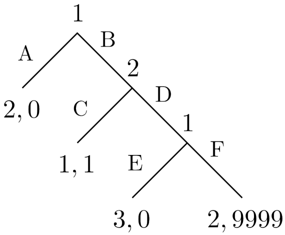


Backward induction

- In the previous two examples, there are a leader and a follower.
- Before the leader can make her decision, she anticipates what the follower will do.
- In general, when there are multiple **stages** in a **dynamic game**, we analyze those decision problems **from the last stage**.
  - Then the second last stage problem can be solved by having the last stage behavior in mind.
  - The the third last stage problem can be solved.
  - We move **backwards** until the first stage problem is solved.
- This solution concept is called **backward induction**.

A three-stage dynamic game

- Consider the three-stage game:
  - In this game, player 1 has two moves: at stage 1 and at stage 3.
  - Player 2 has only one move: at stage 2.
- What will be the equilibrium outcome?



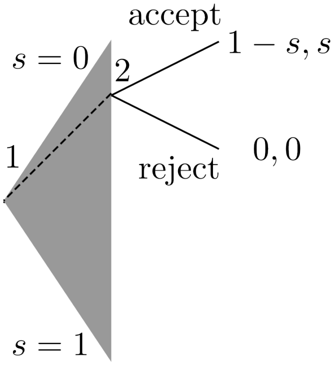
- When player 2 has the chance to act, will she always choose C?
  - If player 1 is **rational**, player 2 should never get a chance to act.
  - If player 2 gets a chance to act, player 1 is somewhat not fully rational.
  - Therefore, if player 2 chooses D, it is **possible** for player 1 to choose F.
  - So player 2 should not completely abandon D.
- **Bounded rationality** has been studied in various subjects.
  - We will not touch it in this course.

Leader's advantage

- In BoS, being the leader (who acts first) is beneficial.
- In prisoners' dilemma, being the leader or not does not matter.
- In most chess games, being the leader is advantageous.
- Is it always good to be the leader?
  - No; the dynamic matching pennies game is an example.

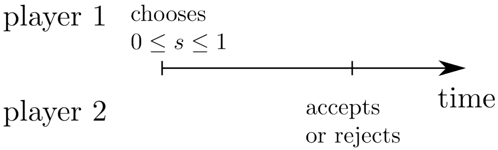
The ultimatum game

- We conclude this section with the classical **ultimatum game**.
  - Player 1 decides how to share \$1 with player 2 by offering him \$$s$.
  - Player 2 may accept or reject the offer.
  - If he accepts, he earns \$$s$ and player 1 earns \$$(1-s)$.
  - If he rejects, both of them earns \$0.
- Suppose both of them are completely rational and want to maximize their payoffs. What will they do?



The time line representation

- In many cases (e.g., when a player has an infinite action space), it is a good idea to use a **time line** to depict the timing of a dynamic game.



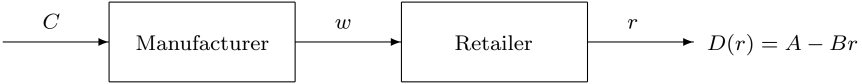
- In equilibrium, player 1 earns \$1 and player 2 earns \$0!
  - In practice, player 1 earns $\$(1-\epsilon)$ and player 2 earns $\$\epsilon$ for some $\epsilon > 0$.
  - Theoretically, however, only (0, accept) and (0, reject) may be equilibrium outcomes.
- This applies to many real-world cases:
  - E.g., wage negotiation between an employer and a employee.
- How may we modify this game to achieve a half-half allocation?

Road map

- Basic ideas.
- **Pricing in a supply chain**.
- Indirect newsvendors.

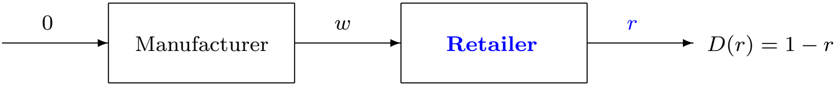
Pricing in a supply chain

- There is a manufacturer and a retailer in a supply chain.



  - The manufacturer produces and supplies to the retailer. The retailer sells to end consumers.
  - The manufacturer sets the **wholesale price w** and then the retailer sets the **retail price r**.
- The demand is $D(r) = A - Br$, where A and B are known constants.
- The unit production cost is C, a known constant.
- What is the equilibrium (i.e., what will the two players do)?
- To make our lives easier, let's assume $A = B = 1$ and $C = 0$.
- Let's apply **backward induction** to solve this game.

The retailer's strategy

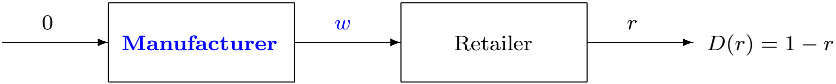


- For the retailer, the wholesale price is **given**. His trade off:
  - Making price lower decreases the profit margin $r - w$.
  - Making price higher decreases the sales volume $1 - r$.
- The retailer's problem:

$$\begin{aligned} & \max\ (r-w)(1-r) \\ = & \max\ -r^2 + (w+1)r - w. \end{aligned}$$

- The optimal solution (best response) is $r^*(w) = \frac{w+1}{2}$.

The manufacturer's strategy



- The manufacturer **predicts** the retailer's decision:
  - Given her offer w, the retail price will be $r^*(w) = \frac{w+1}{2}$.
  - More importantly, the **order quantity** will be

$$1 - r^*(w) = 1 - \frac{w+1}{2} = \frac{1-w}{2}.$$

  - The manufacturer's problem:

$$\max \ w\left(\frac{1-w}{2}\right) = \max \ \frac{-w^2 + w}{2}.$$

  - The optimal solution is $w^* = \frac{1}{2}$.

Equilibrium outcome

- Given that the manufacturer will offer the wholesale price $w^* = \frac{1}{2}$, the resulting retail price will be
$$r^* \equiv r^*(w^*) = \frac{w^* + 1}{2} = \frac{\frac{1}{2} + 1}{2} = \frac{3}{4} > \frac{1}{2} = w^*.$$
  - A common phenomenon called **double marginalization**.
- The **sales volume** is $D(r^*) = 1 - r^* = \frac{1}{4}$.
- The retailer earns
$$(r^* - w^*)D(r^*) = \left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}.$$
- The manufacturer earns
$$(w^* - C)D(r^*) = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}.$$
- In total, they earn
$$\frac{1}{16} + \frac{1}{8} = \frac{3}{16}.$$

Pricing in a cooperative supply chain

- Suppose the two firms **cooperate** and discuss what to do together.
  - They can decide the wholesale and retail prices together.
  - Can they **do better** than when the supply chain is decentralized?
- Let's set $w^{\text{FB}} = 0$:
  - The retailer's best response is

$$r^{\text{FB}} = \frac{1 - w^{\text{FB}}}{2} = \frac{1}{2}.$$

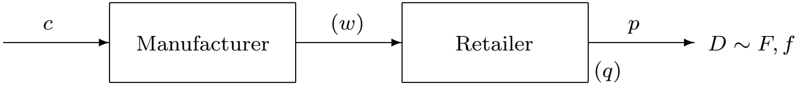
  - The sales volume is $D(r^{\text{FB}}) = 1 - \frac{1}{2} = \frac{1}{2}$.
  - The total profit is $r^{\text{FB}} D(r^{\text{FB}}) = \frac{1}{4}$.
  - This is larger than $\frac{3}{16}$, the total profit generated under decentralization.
- Consumers also benefit from integration.
- However, the manufacturer earns **nothing**.
  - How to make the manufacturer accept the proposal?

Road map

- Basic ideas.
- Pricing in a supply chain.
- **Indirect newsvendors**.

Indirect newsvendor

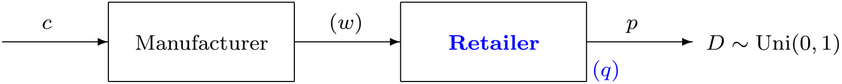
- Consumer demands are not always certain.
- Let's assume that the retailer is a price taker and makes **inventory** decisions for **perishable** products.



- Decisions:
  - The manufacturer chooses the **wholesale price** w.
  - The retailer, facing uncertain demand $D \sim F, f$ and fixed retail price p, chooses the **order quantity** (inventory level) q.
  - Assumption: $D \geq 0$ and is continuous: $F' = f$.
- They try to maximize:
  - Retailer: $\pi_{\mathrm{R}}(q) = p\mathbb{E}[\min\{D, q\}] - wq$.
  - Manufacturer: $\pi_{\mathrm{M}}(w) = (w - c)q^*$, where q^* is optimal to $\max_q\{\pi_{\mathrm{R}}(q)\}$.

Indirect newsvendor with uniform demand

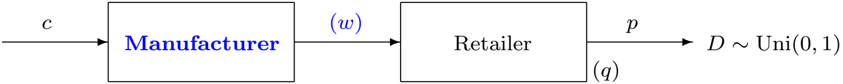
- Suppose the demand is uniformly distributed between 0 and 1.



- The retailer is facing exactly a newsvendor problem:
  - The overage cost is w and the underage cost is $p - w$.
  - The retailer-optimal order quantity $q^*(w)$ satisfies

$$1 - F(q^*(w)) = 1 - q^*(w) = \frac{w}{p} \quad \Leftrightarrow \quad q^*(w) = 1 - \frac{w}{p}.$$

Indirect newsvendor with uniform demand



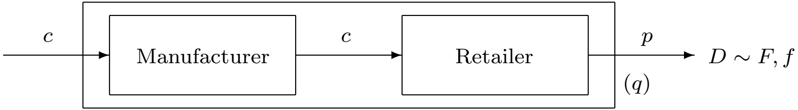
- The manufacturer solves

$$\max_{q} \, (w - c)q^* = (w - c)\left(1 - \frac{w}{p}\right).$$

  - The equilibrium wholesale price is $w^* = \frac{p+c}{2}$.
  - The equilibrium order quantity is $q^* = q^*(w^*) = \frac{p-c}{2p}$.
- What if they cooperate to maximize the aggregate profit?
  - The wholesale price simply determines an **internal transfer**.
  - What matters is the inventory level: $q^{\text{FB}} = 1 - \frac{c}{p} = \frac{p-c}{c}$.
  - As $q^* = \frac{1}{2}q^{\text{FB}}$, decentralization is **inefficient**.
- Is it always the case?

Efficient inventory level in general

- Suppose the two firms integrate:



- They choose q to maximize $\pi_C(q) = p\mathbb{E}[\min\{D, q\}] - cq$.

Proposition 1

The efficient inventory level q^{FB} satisfies $F(q^{FB}) = 1 - \frac{c}{p}$.

Proof. Because $\pi_C(q) = r\{\int_0^q xf(x)dx + \int_q^\infty qf(x)dx\} - cq$, we have $\pi_C'(q) = r[1 - F(q)] - c$ and $\pi_C''(q) = -rf(q) \leq 0$. Therefore, $\pi_C(q)$ is concave and $\pi_C'(q^{\text{FB}}) = 0$ is the given condition. □

Retailer-optimal inventory level

- The retailer maximizes $\pi_{\rm R}(q) = p\mathbb{E}[\min\{D, q\}] - wq$.
- Let q^* be the retailer-optimal inventory level: $\pi_{\rm R}(q^*) \geq \pi_{\rm R}(q)$ for all q.

Proposition 2

We have $q^* < q^{FB}$ if F is strictly increasing.

Proof. Similar to the derivation for $q^{\rm FB}$, we have $F(q^*) = 1 - \frac{w}{p}$ given any wholesale price w. Note that $F(q^*) = 1 - \frac{w}{p} < 1 - \frac{c}{p} = F(q^{\rm FB})$ if $w > c$, which is true in any equilibrium. Therefore, once F is strictly increasing, we have $q^* < q^{\rm FB}$. □

- Decentralization again introduces **inefficiency**.
  - Similar to double marginalization.
  - Does it benefit or hurt consumers?
- Any solution?