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Suggested Solution for Homework 2

Solution providers: Kiwi Liu and Amy Liu
Department of Information Management
National Taiwan University

1. Let the parameters be

$$\begin{aligned}
F_i &= \text{the fixed cost of plant } i,\ i = 1, ..., 3, \\
V_i &= \text{the variable cost of plant } i,\ i = 1, ..., 3, \\
C_i &= \text{the capacity of plant } i,\ i = 1, ..., 3.
\end{aligned}$$

Let the decision variables be

$$\begin{aligned}
x_i &= \text{the amount of production of plant } i,\ i = 1, ..., 3, \\
y_i &= \begin{cases} 1 & \text{if the plat } i \text{ is used to make products} \\ 0 & \text{otherwise} \end{cases},\ i = 1, ..., 3.
\end{aligned}$$

$$\begin{aligned}
\min \quad & \sum_{i=1}^{3} (F_i y_i + V_i x_i) \\
\text{s.t.} \quad & x_i \leq C_i y_i \quad \forall i = 1, ..., 3 \\
& \sum_{i=1}^{3} x_i = 25000 \\
& x_i \geq 0 \quad \forall i = 1, ..., 3 \\
& y_i \in \{0, 1\} \quad \forall i = 1, ..., 3.
\end{aligned}$$

2. (a) Let the parameters be

$$\begin{aligned}
F_{ij} &= \text{the fixed cost of plant } j \text{ that make procduct } i,\ i = 1, 2,\ j = 1, ..., 3, \\
V_{ij} &= \text{the variable cost of plant} j \text{ that make procduct } i,\ i = 1, 2,\ j = 1, ..., 3, \\
C_j &= \text{the capacity of plant } j,\ j = 1, ..., 3, \\
D_i &= \text{the demand of product } i,\ i = 1, ..., 2.
\end{aligned}$$

Let the decision variables be

$$\begin{aligned}
x_{ij} &= \text{the amount of production of plant } j \text{ that make procduct } i,\ i = 1, 2,\ j = 1, ..., 3, \\
y_{ij} &= \begin{cases} 1 & \text{if the plat } j \text{ is used to make product } i \\ 0 & \text{otherwise} \end{cases},\ i = 1, 2,\ j = 1, ..., 3.
\end{aligned}$$

$$\begin{aligned}
\min \quad & \sum_{i=1}^{2} \sum_{j=1}^{3} (F_{ij} y_{ij} + V_{ij} x_{ij}) \\
\text{s.t.} \quad & \sum_{j=1}^{3} x_{ij} = D_i \quad \forall i = 1, 2 \\
& \sum_{i=1}^{2} x_{ij} \leq C_j \quad \forall j = 1, ..., 3 \\
& x_{ij} \leq C_j y_{ij} \quad \forall i = 1, 2,\ j = 1, ..., 3 \\
& x_{ij} \geq 0 \quad \forall i = 1, 2,\ j = 1, ..., 3 \\
& y_i \in \{0, 1\} \quad \forall i = 1, 2,\ j = 1, ..., 3.
\end{aligned}$$

(b) The minimum cost is \$1013000 and our strategy is
(1) Plant 1 produces 20000 units of product 1
(2) Plant 2 produces 5000 units of product 1 and 13000 units of product 2
(3) plant 3 produces 7000 units of product 2

3. (a) The linear relaxation of original LP is

$$\begin{array}{rl} \max & 3x_1 + 4x_2 + 5x_3 + 4x_4 \\ \text{s.t.} & 2x_1 + 3x_2 + 4x_3 + 4x_4 \leq 8 \\ & x_1 \leq 1 \\ & x_2 \leq 1 \\ & x_3 \leq 1 \\ & x_4 \leq 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 4. \end{array}$$

There are four items to select:

Item	1	2	3	4
Value	3	4	5	4
Weight	2	3	4	4
Ratio	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	1

By the tabular, we should put the items in the order: item1 > item2 > item3 > item4. Then, we can get the objective value $10\frac{3}{4}$ and $x^* = (1, 1, \frac{3}{4}, 0)$.

(b) As we know, if a dual constraint is nonbinding, the corresponding primal variable is zero. By Part(a), we know that only $x4$ equals to zero, so the dual constraints (1)(2)(3) must be binding. The dual LP is

$$\begin{array}{rll} \min & 8u_1 + u_2 + u_3 + u_4 + u_5 & \\ \text{s.t.} & 2u_1 + u_2 \geq 3 & (1) \\ & 3u_1 + u_3 \geq 4 & (2) \\ & 4u_1 + u_4 \geq 5 & (3) \\ & 4u_1 + u_5 \geq 4 & \\ & u_i \geq 0 \quad \forall i = 1, ..., 5. & \end{array}$$

(c) By Part(b), we can replace u_2, u_3, u_4 by u_1. Notice that u_1 must be lower than $\frac{5}{4}$ (u_1 should be lower than $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$ due to the replacement of u_2, u_3, u_4). The new dual LP becomes

$$\begin{array}{rl} \min & 12 - u_1 + u_5 \\ \text{s.t.} & 4u_1 + u_5 \geq 4 \\ & u_1 \leq \frac{5}{4} \\ & u_i \geq 0 \quad \forall i = 1,\ 5. \end{array}$$

The feasible region and isoquant line are illustrated in Figure 1. It is clear that we should push the isoquant line until we stop at the extreme point ($\frac{5}{4}$, 0), which is an optimal solution and we get objective value $10\frac{3}{4}$.

(d) Yes. We get the same objective value in Part(a) and (c), and x^* and u^* are also primal and dual feasible, so x^* and u^* are primal and dual optimal due to the strong duality.

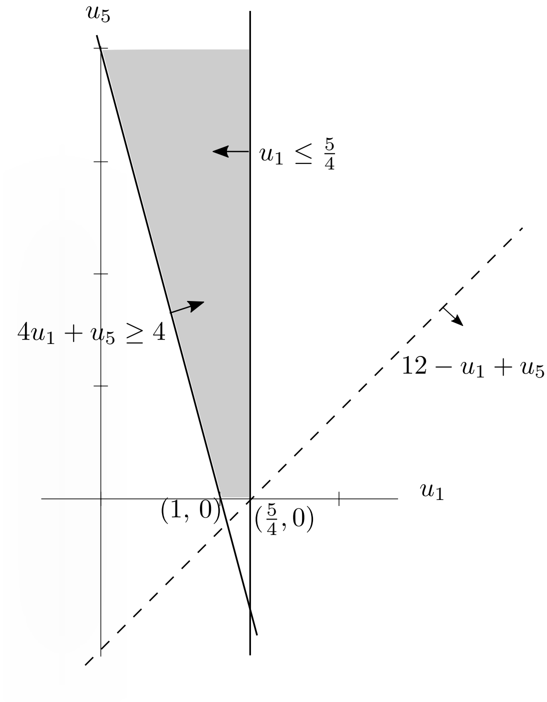

Figure 1: Graphical solution of the remaining dual LP in Problem 3

4. The branch-and-bound tree for solving this problem is depicted in Figure 2. The optimal solution is (0,0,1,1) or (0,1,0,1). The optimal objective value is 9.

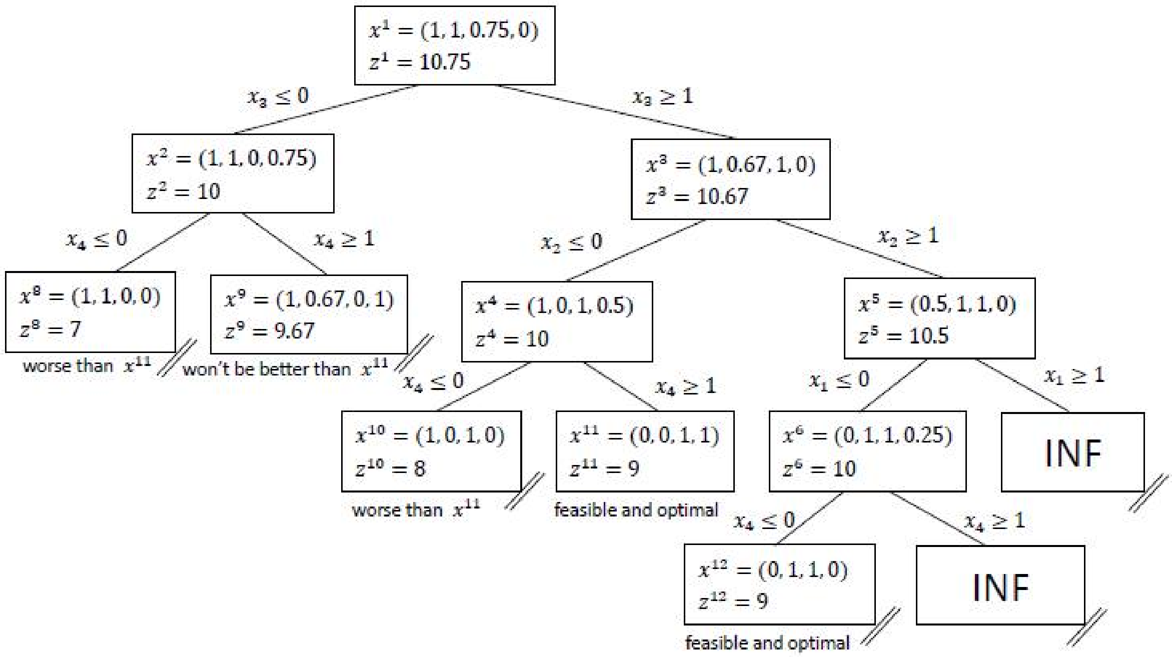

Figure 2: Branch-and-bound tree for Problem 4

5. (a) Let the parameters be

$$w_i = \text{the weight of node } i,\ i = 1, ..., 3,$$

Let the decision variables be

$$x_i = \begin{cases} 1 & \text{if the node } i \text{ is selected} \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 3.$$

$$\begin{aligned} \min \quad & \sum_{i=1}^{3} w_i x_i \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 3. \end{aligned}$$

(b) The linear relaxation of original IP is

$$\begin{aligned} \min \quad & \sum_{i=1}^{3} w_i x_i \\ \text{s.t.} \quad & x_1 + x_2 \geq 1 \\ & x_2 + x_3 \geq 1 \\ & x_1 + x_3 \geq 1 \\ & x_i \in [0, 1] \quad \forall i = 1, ..., 3. \end{aligned}$$

We want to minimize the total weight, and because w_i is given, we would try to minimize x_i. If $x_i > 1$, it will definitely satisfy the constraint $x_i + x_v \geq 1$. Let $x_i^{'} = 1$. Then the constraint becomes $(1 + x_v) \geq 1$ and is always satisfied. Because minimizing x_i to 1 increases less cost, we always want to let $x_i = 1$.

(c) Let x_4, x_5, x_6 be the slack variables. The standard form of (p) is

$$\begin{aligned} \min \quad & \sum_{i=1}^{3} w_i x_i \\ \text{s.t.} \quad & x_1 + x_2 - x_4 = 1 \\ & x_2 + x_3 - x_5 = 1 \\ & x_1 + x_3 - x_6 = 1 \\ & x_i \in [0, 1] \quad \forall i = 1, ..., 6. \end{aligned}$$

We can see that there exist some square submatrices whose determinant isn't 1 or -1, so the coefficient matrix is not totally unimodular.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

(d) The reduced costs are

$$\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T = \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 2 & -1 & -1 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -3 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -3 & -2 \end{bmatrix}$$

We can see that the reduced costs are all negative with the solution x^*. No variable should enter. The solution x^* is optimal.

(e) No, the proposition that total unimoudularity will gives a integer solution doesn't mean that having a integer solution implies the total unimodularity.

(f) If we assign $w_1 = 2$, $w_2 = 2$, $w_3 = 1$, then the LR-optimal solution is $x^* = (0.5, 0.5, 0.5)$. Because the coefficients are not integer, it's not IP-feasible.

(g) We get the feasible solution when we solve the first node by the branch-and-bound tree, so we don't need to draw the tree. The optimal solution is $x^* = (1, 1, 0)$. The optimal objective value is 5.

6. (a) The model is

$$\begin{aligned} \min \quad & \sum_{i \in V} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \forall [i,j] \in E \\ & x_i \in \{0, 1\} \quad \forall i \in V. \end{aligned}$$

(b) The linear relaxation of original IP is

$$\begin{aligned} \min \quad & \sum_{i \in V} w_i x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \quad \forall [i,j] \in E \\ & x_i \in [0, 1] \quad \forall i \in V. \end{aligned}$$

The dual LP is

$$\begin{aligned} \max \quad & \sum_{e \in E} Y_e \\ \text{s.t.} \quad & \sum_{e:[u,i] \in E} Y_e \leq w_i \quad \forall i \in V \\ & Y_e \geq 0 \quad \forall e \in E. \end{aligned}$$

(c) As the above, we can see that the number of primal LP's constraints is equal to the number of edges. On the other hand, the number of dual LP's constraints is equal to the number of nodes.
Let the number of constraints in primal LP denotes M and the number of variables denotes N. We know that when $M < N$, primal LP will be easier to solve; otherwise, the dual LP is easier to solve. Generally, the number of edges is larger than the number of nodes. As the result, the dual LP of the vertex cover problem is easier to solve.