GMBA 7098: Statistics and Data Analysis (Fall 2014)

Estimations

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Road map

- **Statistical estimation**.
- Estimating population mean with known variance.
- Estimating population mean with unknown variance.

Example: average daily consumers

- A retail chain of 3000 stores is going to have a special discount on the next Monday. The manager wants to know the **average number of daily consumers** entering the stores on that day.
- She decides to do a survey on the next Monday.
  - On that day, there will be some consumers entering each store.
  - For store i, $i = 1, ..., 3000$, let x_i be the number of consumers.
  - It is too costly to collect all x_is and calculate $\mu = \frac{\sum_{i=1}^{3000} x_i}{3000}$.
  - This is a task of **estimating** a **parameter**.
- Her budget is enough for hiring 7 temporary workers to count the number of consumers throughout the day.
  - She decides to randomly draw 7 stores and calculate $\bar{x} = \frac{\sum_{i=1}^{7} x_i}{7}$.

Example: average daily consumers

- On that day, she gets the following sample data:
  - She gets 1026, 932, 852, 1212, 844, 822, and 1032 consumers.
  - The **sample mean** is $\bar{x} = 960$.
- Intuitively, she will think that the population mean μ is "around" 960.
- Suppose she concludes that "μ is within 950 and 970," how much confidence may she have?
- In general, is it okay to conclude that $\mu \in [\bar{x} - 10, \bar{x} + 10]$?

Estimations

- One of the most important statistical tasks is **estimation**.
  - For unknown population **parameters**, we estimate them through **statistics** obtained from samples.
  - For example, when the population mean is unknown, we use sample mean as an estimate.
- We want to go beyond intuitions and conjectures.
  - We need some knowledge about the **sampling distributions**.
  - E.g., we know $\overline{X} \sim \text{ND}(\mu, \frac{\sigma}{\sqrt{n}})$.
- In statistics, use **confidence intervals** to estimate parameters.
- Let's start from estimating the population mean.

Road map

- Interval estimation.
- **Estimating population mean with known variance**.
- Estimating population mean with unknown variance.

Drawbacks of point estimation

- We may use the sample mean $\bar{x}$ to estimate the population mean μ.
  - "μ should somewhat be close to $\bar{x}$."
  - This is called a **point estimation**.
- However, there are some drawbacks of point estimation:
  - We know that μ is close to $\bar{x}$. But **how close**?
  - What is $|\mu - \bar{x}|$?
  - As μ is unknown, we will never know the answer!
- Instead of suggesting a number, we will suggest an **interval**.
  - Then we measure how good the suggested interval is.

Interval estimation: the first illustration

- Consider a population with unknown μ. For simplicity, let's assume:
  - The population variance σ^2 is **known**.
  - The population follows a **normal** distribution.
- Let the sample mean $\overline{X}$ be the **estimator**.
  - Before sampling, we do not know what will be the sample mean's value.
  - The random variable sample mean is denoted as $\overline{X}$.
  - After sampling, the realized value of the sample mean is $\bar{x}$.
- Suppose $\sigma^2 = 16$ and the sample size $n = 8$.
- Based on $\overline{X}$, we will choose a number b and claim that μ lies in the **interval** $[\overline{X} - b, \overline{X} + b]$.
  - We may be either right or wrong.
  - When b increases, we are more confident that we will be right.
  - However, a larger interval means that the estimation is less accurate.
  - What is the **probability** that we are right?

The sampling distribution

- Question: For any given b, find

$$\Pr\left(\overline{X} - b \leq \mu \leq \overline{X} + b\right).$$

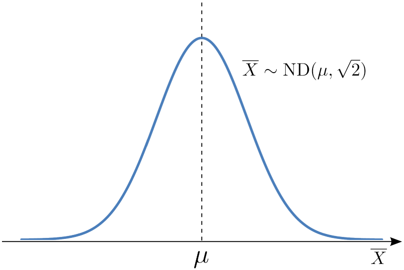
- As the population is normal:

$$\overline{X} \sim \text{ND}\left(\mu, \frac{\sigma}{\sqrt{n}} = \frac{4}{\sqrt{8}} = \sqrt{2}\right).$$

- Suppose someone proposes to set $b = \sqrt{2}$, then the interval will be

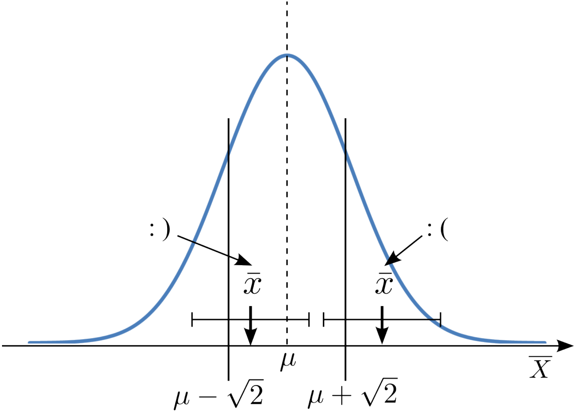
$$\left[\overline{X} - \sqrt{2}, \overline{X} + \sqrt{2}\right].$$

How good the interval is?



How good an interval is?

- If, luckily, $\bar{x}$ is close enough to μ, $[\bar{x} - \sqrt{2}, \bar{x} + \sqrt{2}]$ covers μ.
- If, unluckily, $\bar{x}$ is far from μ, $[\bar{x} - \sqrt{2}, \bar{x} + \sqrt{2}]$ does not cover μ.

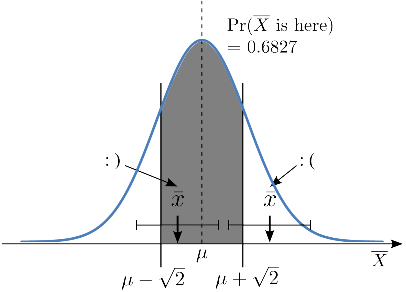


How good an interval is?

- The probability that "we are lucky" can be calculated!
- No matter where μ is, we have

$$\begin{aligned}&\Pr\left(\overline{X}-\sqrt{2}\le\mu\le\overline{X}+\sqrt{2}\right)\\ =&\Pr\left(\mu-\sqrt{2}\le\overline{X}\le\mu+\sqrt{2}\right)\\ =&\,0.6827.\end{aligned}$$

- To calculate this, try `pnorm(mu - sqrt(2), mu, sqrt(2))` for different values of `mu`. You will always see 0.1587.

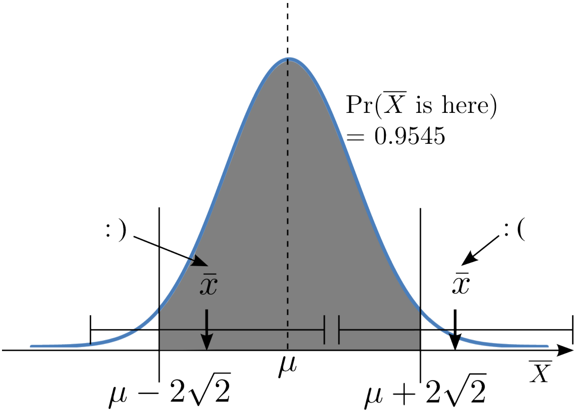


A short summary

- Given **any** realization $\bar{x}$, $[\bar{x} - \sqrt{2}, \bar{x} + \sqrt{2}]$ may or may not covers μ.
- Regarding the random $\overline{X}$, we know $[\overline{X} - \sqrt{2}, \overline{X} + \sqrt{2}]$ covers μ with probability 0.6827.
  - This level of confidence can be calculated as we know $\overline{X} \sim \text{ND}(\mu, \sqrt{2})$.
- Instead of having $\sqrt{2}$ as the leg length, let's try $2\sqrt{2}$.

A larger interval

- The probability that "we are lucky" now becomes 0.9545!
  - $\Pr\left(\overline{X} - 2\sqrt{2} \leq \mu \leq \overline{X} + 2\sqrt{2}\right) = \Pr\left(\mu - 2\sqrt{2} \leq \overline{X} \leq \mu + 2\sqrt{2}\right) = 0.9545.$



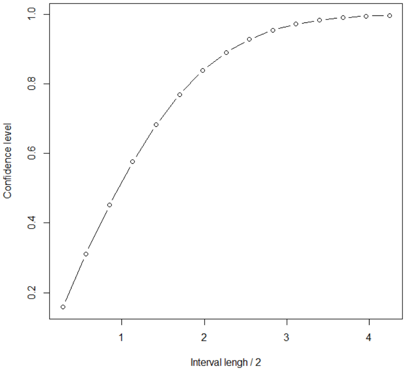
Confidence levels and confidence intervals

- We made two attempts:
  - $[\mu - \sqrt{2}, \mu + \sqrt{2}]$ results in a covering probability 0.6827.
  - $[\mu - 2\sqrt{2}, \mu + 2\sqrt{2}]$ results in another covering probability 0.9545.
- In statistics, when we do interval estimation:
  - Such a "covering probability" is called **confidence level**.
  - These intervals are called **confidence intervals** (CI).
- How to choose the interval length?
  - A larger confidence interval results in a higher confidence.
  - There is a **trade-off** between accurate estimation and high confidence.

Confidence levels vs. interval lengths

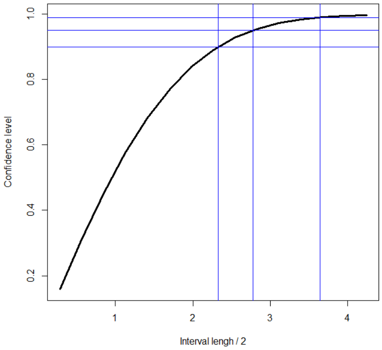
- To find the relationship:

```
z <- seq(0.2, 3, 0.2)
conf <- 1 - 2 * pnorm(-z * sqrt(2), 0, sqrt(2))
plot(z * sigma.xbar, conf, type = "b")
```



How to choose the interval length?

- In practice, we first choose a confidence level, and then we choose the smallest interval that achieves this level.
  - We typically denote the error probability as α.
  - The confidence level is thus $1 - \alpha$.
  - Common confidence levels: 90%, 95%, and 99%.
- How to calculate the leg length?
  - 90%: `-qnorm(0.05, 0, sqrt(2))`.
  - 95%: `-qnorm(0.025, 0, sqrt(2))`.
  - 99%: `-qnorm(0.005, 0, sqrt(2))`.
- Why?



Example revisited: average daily consumers

- Recall that we have 3000 stores, each with a number of consumers on a given day.
  - The population consists of 3000 numbers.
  - There is a population mean μ, which is unknown.
- We collected data from 7 stores:
  - The sample data: 1026, 932, 852, 1212, 844, 822, and 1032.
  - The **sample mean** is $\bar{x} = 960$.
- How to do interval estimation with this sample?

Conducting the estimation

- We must know the population variance σ^2.
  - Let's assume that $\sigma = 120$.
- We need either the population is normal or the sample size is large.
  - Let's assume that the population is normal.
- Now we are ready to construct a confidence interval. Let's construct three intervals for $1 - \alpha = 0.9$, 0.95, and 0.99.
  - Step 1: $\bar{x} = 960$.
  - Step 2: The standard deviation of the sample mean is $\frac{\sigma}{\sqrt{n}} = 45.356$.[1]
  - Step 3: The leg lengths are **`-qnorm(0.05, 0, 45.356)`**, **`-qnorm(0.025, 0, 45.356)`**, and **`-qnorm(0.005, 0, 45.356)`**, which are 74.604, 88.896, and 116.829.
  - Step 4: The interval with 90% confidence level is $[960 - 74.604, 960 + 74.604] = [885.39, 1034.60]$. The other two intervals are $[871.10, 1048.90]$ and $[843.171076.82]$.

[1] This quantity $\frac{\sigma}{\sqrt{n}}$ is called the **standard error** of the sample mean.

Interpreting the estimation

- Consider the interval with 95% confidence level: $[871.10, 1048.90]$.
- What is the business implication?
  - We will claim that the true average daily consumers for all the 3000 stores is within 870 and 1050.
  - We are 95% confident. It is quite unlikely for us to be wrong.
- Recall that there is a special discount on that day.
  - Suppose the marketing manager has promised that "the average daily consumers will be at least 850."
  - Now we have a strong evidence showing that the target is really achieved.
- Note that maybe in fact $\mu < 850$. We are just "quite confident."

Summary

- Facing an unknown population mean μ (with a known population variance σ^2), we may construct a confidence interval:
  - Centered at the to-be-realized sample mean $\overline{X}$.
  - Will cover μ with a predetermined probability.
- We need one of the following:
  - The population follows a normal distribution.
  - The sample size $n \geq 30$.

Road map

- Interval estimation.
- Estimating population mean with known variance.
- **Estimating population mean with unknown variance**.

Estimation without the population variance

- Sometimes (actually for most of the time) we **do not** know the population variance σ^2.
- Then we cannot calculate the standard error $\frac{\sigma}{\sqrt{n}}$.
- In this case, intuitively we may try to replace σ by s, the **sample standard deviation**.
  - As an example, for the 7 numbers of consumers 1026, 932, 852, 1212, 844, 822, and 1032, we have[2]

$$s = \sqrt{\frac{(1026-960)^2 + \cdots + (1032-960)^2}{7-1}} = 140.233.$$

  - We then use $\frac{s}{\sqrt{n}}$ to construct an interval.
  - However, $\overline{X} \sim \text{ND}(\mu, \frac{s}{\sqrt{n}})$ is not right!
- We need some adjustments.

[2] The R function `sd()` can do all the calculations for you.

The t distribution

- Let S be the sample standard deviation (which is random before sampling) and s be its realization.
- When we replace σ by S, we rely on the following fact:

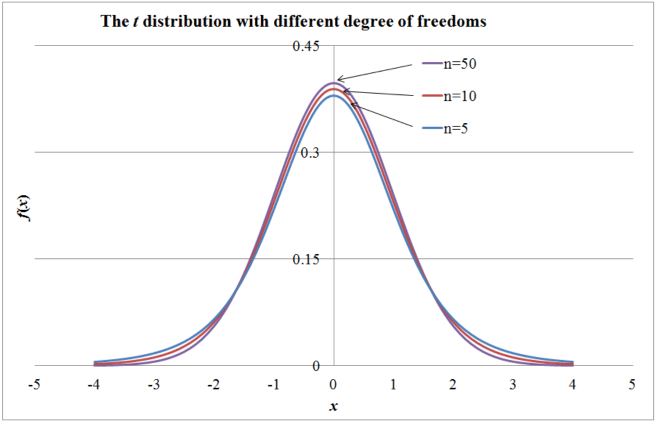
Proposition 1

For a normal population, the quantity $T = \frac{\overline{X} - \mu}{S/\sqrt{n}}$ follows the t distribution with degree of freedom $n - 1$.

  - We know the sampling distribution of T (when the population is normal). We call it **the t distribution**.
  - The only parameter is the **degree of freedom**, which is $n - 1$.
  - Its probability density function is known (and we do not care about it!) Relevant probabilities may be calculated with software.
  - In R, we use the functions `pt(q, df)` and `qt(p, df)`.

The t distributions

- The t distribution is **symmetric**, **centered at 0**, and **bell-shaped**.
- When n goes up, it approaches the **standard normal distribution**.



The history of the t distribution

- The t distribution is also called the **Student's t distribution**.
  - The author William Gosset's company forbids employees from publishing their findings.

VOLUME VI — MARCH, 1908 — No. 1

BIOMETRIKA.

THE PROBABLE ERROR OF A MEAN.

BY STUDENT.

Introduction.

ANY experiment may be regarded as forming an individual of a "population" of experiments which might be performed under the same conditions. A series of experiments is a sample drawn from this population.

(http://www.aliquote.org/cours/2012_biomed/biblio/Student1908.pdf)

Applying the t distribution

- Before sampling, we know we will get the sample mean $\overline{X}$ and sample standard deviation S.
- For any b, we construct an interval $[\overline{X}-b, \overline{X}+b]$. We want to know $\Pr(\overline{X}-b \leq \mu \leq \overline{X}+b)$.
- Now we do not know the distribution of $\overline{X}$; we only know the distribution of $T = \frac{\overline{X}-\mu}{S/\sqrt{n}}$. Therefore:

$$\Pr\left(\overline{X}-b \leq \mu \leq \overline{X}+b\right) = \Pr\left(\mu - b \leq \overline{X} \leq \mu + b\right)$$
$$= \Pr\left(\frac{-b}{S/\sqrt{n}} \leq \frac{\overline{X}-\mu}{S/\sqrt{n}} \leq \frac{b}{S/\sqrt{n}}\right) = \Pr\left(\frac{-b}{S/\sqrt{n}} \leq T \leq \frac{b}{S/\sqrt{n}}\right).$$

- Once we obtain s, we may calculate the probability.

Applying the t distribution

- Consider the example of estimating average daily consumers again.
- Suppose we do not know the population variance σ^2.
  - We know $\bar{x} = 960$ and $s = 140.233$.
- Suppose we propose the interval $[860, 1060]$.
  - We calculate

$$\Pr\left(\frac{-b}{S/\sqrt{n}} \leq T \leq \frac{b}{S/\sqrt{n}}\right) = \Pr\left(\frac{-100}{140.233/\sqrt{7}} \leq T \leq \frac{100}{140.233/\sqrt{7}}\right)$$
$$= \Pr(-1.887 \leq T \leq 1.887) = 0.892,$$

where the last step is done with `1 - 2 * pt(-1.887, 6)`.

- We are 89.2% confident that the average number of daily consumers lies within 860 and 1060.

From a confidence level to an interval

- How to construct an interval $[\overline{X} - b, \overline{X} + b]$ for us to be 95% confident?
- We have the t distribution; given any value t, we know $\Pr(T \leq t)$.
  - When the degree of freedom is 6, $\Pr(T \leq -2.447) = 0.025$.
  - Use `qt(0.025, 6)`!
- Moreover, we have

$$\Pr(T \leq t) = \Pr\left(\frac{\overline{X} - \mu}{S/\sqrt{n}} \leq t\right) = \Pr\left(\mu \geq \overline{X} - t\frac{S}{\sqrt{n}}\right).$$

- We need the leg length to be $-t\frac{S}{\sqrt{n}} = 2.447 \times \frac{140.233}{\sqrt{7}} = 129.694$.
  - The multiplier $\frac{S}{\sqrt{n}}$ will always be used.
- The desired interval is

$$[960 - 129.694, 960 + 129.694] = [885.40, 1034.60].$$

From a confidence level to an interval

- In general, given $\bar{x}$, s, n, and α, we construct the confidence interval in the following steps:
  - Step 1: Calculate the multiplier $\frac{s}{\sqrt{n}}$.
  - Step 2: Calculate the **critical value** t^* as `-qt(p, df)`, where `p` is $\frac{\alpha}{2}$ and `df` is $n-1$.
  - Step 3: The product of the critical t^* and multiplier $\frac{s}{\sqrt{n}}$ is the leg length.
  - Step 4: The interval is $[\bar{x} - t^* \frac{s}{\sqrt{n}}, \bar{x} + t^* \frac{s}{\sqrt{n}}]$.

Comparing the two situations

- σ^2 may be known or unknown:
  - With σ^2, we know $\overline{X} \sim \text{ND}(\mu, \frac{\sigma}{\sqrt{n}})$, and thus $Z = \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim \text{ND}(0,1)$, the standard normal distribution, or the z **distribution**.
  - Without σ^2, $T = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(n-1)$, the t distribution.
- With σ^2, the confidence interval is $[\bar{x} - z^* \frac{\sigma}{\sqrt{n}}, \bar{x} + z^* \frac{\sigma}{\sqrt{n}}]$,
  - The **critical value** z^* is calculated as `-qnorm(p, 0, 1)`, or simply `-qnorm(p)`, with `p` equals $\frac{\alpha}{2}$.
- Conclusion:
  - With σ^2, the leg length is $z^* \frac{\sigma}{\sqrt{n}}$; use `-qnorm(p)` to find z^*.
  - Without σ^2, the leg length is $t^* \frac{s}{\sqrt{n}}$; use `-qt(p)` to find t^*.
  - The left-tail probability `p` is $\frac{\alpha}{2}$.

Remarks

- If the population is normal, the sample size n does not matter.
  - We may use the t distribution anyway.
- If the population is **non-normal** and the sample size is large ($n \geq 30$):
  - The population is non-normal, so we cannot use the t distribution.
  - The sample size is large, so according to the **central limit theorem**, the sample mean is normal.
  - For $n \geq 30$, $t(n-1)$ is very close to z.
  - Using the t distribution as an approximation is acceptable.
- If the population is non-normal and the sample size is small ($n < 30$), using t distribution for estimation is inaccurate.
  - However, the t distribution for estimating the population mean is **robust** to the normal population assumption: Having nonnormal population does not harm a lot.
  - We still suggest one not to use the t distribution in this case.

Summary

- To estimate the population mean μ:

σ^2	Sample size	Population distribution: Normal	Population distribution: Nonnormal
Known	$n \geq 30$	z	z
	$n < 30$	z	Nonparametric
Unknown	$n \geq 30$	t	t
	$n < 30$	t	Nonparametric

  - Nonparametric methods are beyond the scope of this course.