Statistics and Data Analysis

Hypothesis Testing

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Introduction

- How do **scientists** (physicists, chemists, etc.) do research?
  - Observe phenomena.
  - Make hypotheses.
  - Test the hypotheses through experiments (or other methods).
  - Make conclusions about the hypotheses.
- In the business world, business researchers do the same thing with **hypothesis testing**.
  - One of the most important technique of statistical inference.
  - A technique for (statistically) **proving** things.
  - Again relies on **sampling distributions**.

Road map

- **Basic ideas of hypothesis testing**.
- The first example.
- The p-value.

People ask questions

- In the business (or social science) world, people ask questions:
  - Are older workers more loyal to a company?
  - Does the newly hired CEO enhance our profitability?
  - Is one candidate preferred by more than 50% voters?
  - Do teenagers eat fast food more often than adults?
  - Is the quality of our products stable enough?
- How should we answer these questions?
- Statisticians suggest:
  - First make a **hypothesis**.
  - Then **test** it with samples and statistical methods.

Statistical hypotheses

- A **statistical hypothesis** is a formal way of stating a hypothesis.
  - Typically it is a mathematical description of parameters to test.
- It contains two parts:
  - The **null hypothesis** (denoted as H_0).
  - The **alternative hypothesis** (denoted as H_a or H_1).
- The alternative hypothesis is:
  - The thing that we want (need) to prove.
  - The conclusion that can be made only if we have **a strong evidence**.
- The null hypothesis corresponds to a **default** position.
  - We first **assume** that the null hypothesis is correct.
  - Then we collect sample data.
  - If under the null hypothesis it is **quite unlikely** to see our observed result, we claim that the null hypothesis is wrong.

Statistical hypotheses: example 1

- In our factory, we produce packs of candy whose average weight should be 1 kg.
- One day, a consumer told us that his pack only weighs 900 g.
- We need to know whether this is just a rare event or our production system is out of control.
- If (we believe) the system is out of control, we need to shutdown the machine and spend two days for inspection and maintenance. This will cost us at least $100,000.
- So we should not to believe that our system is out of control just because of one complaint. What should we do?

Statistical hypotheses: example 1

- We first state a hypothesis: "Our production system is under control."
- Then we ask: Is there a strong enough evidence showing that the hypothesis is **wrong**, i.e., the system is out of control?
  - Initially, we assume that our system is under control.
  - Then we do a survey to see if we have a strong enough evidence.
  - We shutdown machines **only if** we can "prove" that the system is indeed out of control.
- Let μ be the average weight, the **statistical hypothesis** is

$$
\begin{aligned}
\mathrm{H}_0 &: \mu = 1 \\
\mathrm{H}_a &: \mu \neq 1.
\end{aligned}
$$

Statistical hypotheses: example 2

- In our society, we adopt the presumption of innocence.
  - One is considered **innocent** until proven **guilty**.
- So when there is a person who probably stole some money:

$$\begin{aligned} \mathrm{H}_0 &: \text{The person is innocent} \\ \mathrm{H}_\mathrm{a} &: \text{The person is guilty.} \end{aligned}$$

- There are two possible errors:
  - One is guilty but we think she/he is innocent.
  - One is innocent but we think she/he is guilty.
- Which one is more critical?
  - It is unacceptable that an innocent person is considered guilty.
  - We will say one is guilty **only if** there is a strong evidence.

Statistical hypotheses: example 3

- Consider the following hypothesis: "The candidate is preferred by more than 50% voters."
- As we need a default position, and the percentage that we care about is 50%, we will choose our null hypothesis as

$$\mathrm{H}_0 : p = 0.5.$$

  - p is the **population proportion** of voters preferring the candidate.
  - More precisely, let $X_i = 1$ if voter i prefers this candidate and 0 otherwise, $i = 1, ..., N$, then $p = \frac{\sum_{i=1}^{N} X_i}{N}$.
- How about the alternative hypothesis? Should it be

$$\mathrm{H}_\mathrm{a} : p > 0.5 \quad \text{or} \quad \mathrm{H}_\mathrm{a} : p < 0.5?$$

Statistical hypotheses: example 3

- The choice of the alternative hypothesis depends on the related **decisions** or **actions** to make.
- Suppose one will go for the election only if she thinks she will win (i.e., $p > 0.5$), the alternative hypothesis will be

$$\mathrm{H_a}: p > 0.5.$$

- Suppose one tends to participate in the election and will give up only if the chance is slim, the alternative hypothesis will be

$$\mathrm{H_a}: p < 0.5.$$

- The alternative hypothesis is "the thing we want (need) to prove."

Remarks

- For setting up a statistical hypothesis:
  - Our default position will be put in the null hypothesis.
  - The thing we want to prove (i.e., the thing that needs a strong evidence) will be put in the alternative hypothesis.
- For writing the mathematical statement:
  - The **equal sign** (=) will always be put in the null hypothesis.
  - The alternative hypothesis contains an **unequal sign** or **strict inequality**: $\neq$, $>$, or $<$.
- The direction of the alternative hypothesis, when it is an inequality, depends on the business context.

One-tailed tests and two-tailed tests

- If the alternative hypothesis contains an unequal sign ($\neq$), the test is a **two-tailed** test.
- If it contains a strict inequality ($>$ or $<$), the test is a **one-tailed** test.
- Suppose we want to test the value of the population mean.
  - In a two-tailed test, we test whether the population mean significantly deviates from a hypothesized value. We do not care whether it is larger than or smaller than.
  - In a one-tailed test, we test whether the population mean significantly deviates from a hypothesized value **in a specific direction**.

Road map

- Basic ideas of hypothesis testing.
- **The first example**.
  - **A two-tailed test**.
  - A one-tailed test.
- The p-value.

The first example: a two-tailed

- Now we will demonstrate the process of hypothesis testing.
- Suppose we test the average weight (in g) of our products.

$$\begin{aligned} \mathrm{H}_0 &: \mu = 1000 \\ \mathrm{H}_a &: \mu \neq 1000. \end{aligned}$$

- The variance of the product weights is $\sigma^2 = 40000$ g^2.
  - The case with unknown σ^2 will be discussed in the next lecture.
- A random sample has been collected.
  - Suppose the sample size $n = 100$.
  - Suppose the sample mean $\bar{x} = 963$.
- How to make a conclusion?

Controlling the error probability

- All we can do is to collect a **random** sample and make our conclusion based on the observed sample.
- It is natural that we may be wrong when we claim $\mu \neq 1000$.
  - It is possible that $\mu = 1000$ but we unluckily get a sample mean $\bar{x} = 812$.
- We want to **control the error probability**.
  - Let α be the maximum probability for us to make this error.
  - α is called the **significance level**.
  - $1 - \alpha$ is called the **confidence level**.
  - Target: If $\mu = 1000$, our sampling and testing process will make us claim that $\mu \neq 1000$ with probability at most α.

Rejection rule

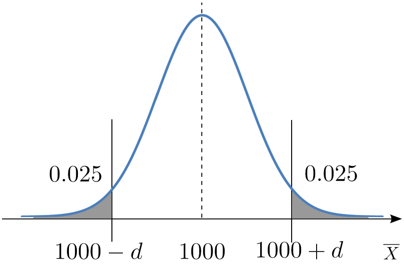
- Now let's test with the significance level $\alpha = 0.05$.
- Intuitively, if $\overline{X}$ **deviates** from 1000 **a lot**, we should **reject** the null hypothesis and believe that $\mu \neq 1000$.
  - If $\mu = 1000$, it is so unlikely to observe such a large deviation.
  - So such a large deviation provides a **strong evidence**.
- So we start by sampling and calculating the **sample mean**.
- We want to construct a **rejection rule**: If $|\overline{X} - 1000| > d$, we reject H_0. We need to calculate d.

Rejection rule

- We want a distance d such that **if H_0 is true**, the probability of rejecting H_0 is 5%, i.e.,

$$\Pr\left(|\overline{X} - 1000| > d \middle| \mu = 1000\right) = 0.05.$$

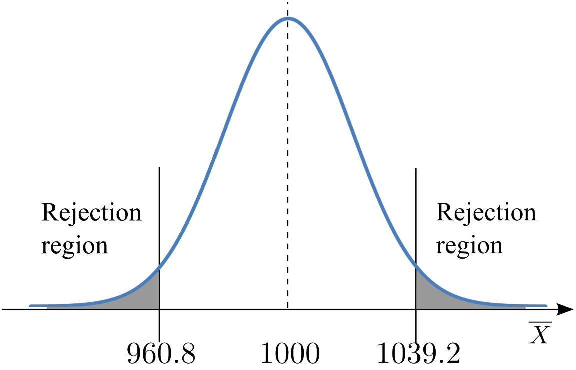
  - People typically hide the condition $\mu = 1000$ and directly write $\Pr(|\overline{X} - 1000| > d)$.
- Consider $\overline{X}$:
  - We know $\sigma = 200$ and $n = 100$.
  - We **assume** that $\mu = 1000$.
  - Thanks to the central limit theorem, $\overline{X} \sim \text{ND}(1000, 20)$.



$$\Pr(|\overline{X} - 1000| > d) = 0.05.$$

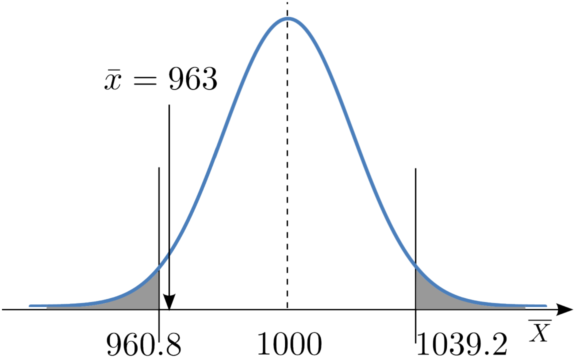
Rejection rule: the critical value

- According to $\overline{X} \sim \text{ND}(1000, 20)$, $\Pr(|\overline{X} - 1000| > 39.2) = 0.05$. The **rejection region** is $R = (-\infty, 960.8) \cup (1039.2, \infty)$.
- If $\overline{X}$ falls in the rejection region, we reject H_0.



Rejection rule: the critical value

- Because $\bar{x} = 963 \notin R$, we **cannot reject $\mathbf{H}_0$**.
  - The deviation from 1000 is not large enough.
  - The evidence is not strong enough.



Rejection rule: the critical value

- In this example, the two values 960.8 and 1039.2 are the **critical values** for rejection.
  - If the sample mean is more extreme than one of the critical values, we reject H_0.
  - Otherwise, we do not reject H_0.
- $\bar{x} = 963$ is not strong enough to support H_a: $\mu \neq 1000$.
- Concluding statement:
  - Because the sample mean does not lie in the rejection region, we **cannot reject H_0**.
  - With a 95% confidence level, there is **no** strong evidence showing that the average weight **is not** 1000 g.
  - Therefore, we **should not** shutdown machines to do an inspection.

Summary

- We want to know whether the machine is out of control.
  - If the machine is actually good, we do not want to reach a conclusion that requires an inspection and maintenance.
  - We will do the inspection **only if** we have a strong evidence suggesting that $\mu \neq 1000$.
- We want to know whether H_0 is false, i.e., $\mu \neq 1000$.
- We control the probability of making a wrong conclusion.
  - We should not reject H_0 if it is true,
  - We limit the probability at $\alpha = 5\%$.
- We will conclude that H_0 is false if $\overline{X}$ falls in the rejection region.
  - The calculation of the the critical values is based on the normal distribution, which can always be transformed to **the z distribution**.
  - This is called a **z test**.

Not rejecting vs. accepting

- We should be careful in writing our conclusions:
  - **Wrong**: Because the sample mean does not lie in the rejection region, we **accept** H_0. With a 95% confidence level, there **is** a strong evidence showing that the average weight **is** 1000 g.
  - **Right**: Because the sample mean does not lie in the rejection region, we **cannot reject H_0**. With a 95% confidence level, there **is no** strong evidence showing that the average weight **is not** 1000 g.
  - Unable to prove one thing is false does not mean it is true!

Road map

- Basic ideas of hypothesis testing.
- **The first example**.
  - A two-tailed test.
  - **A one-tailed test**.
- The p-value.

The first example (part 2)

- Suppose that we modify the hypothesis into a directional one:[1]

$$\begin{aligned} \mathrm{H}_0 &: \mu = 1000. \\ \mathrm{H}_\mathrm{a} &: \mu < 1000. \end{aligned}$$

We still have $\sigma^2 = 40000$, $n = 100$, and $\alpha = 0.05$.

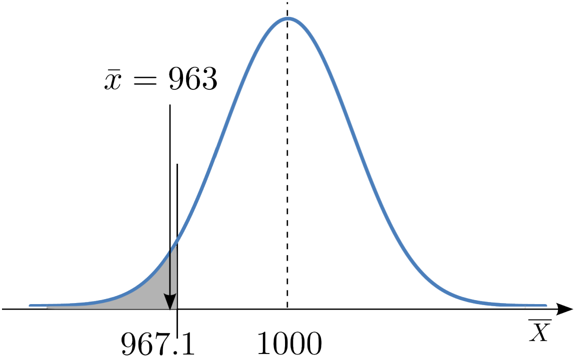
- This is a **one-tailed test**.
- Once we have a strong evidence supporting H_a, we will claim that $\mu < 1000$.
- We need to find a distance d such that

$$\Pr\left(1000 - \overline{X} > d \middle| \mu = 1000\right) = 0.05.$$

[1] Some researchers write $\mu \geq 1000$ in this case.

Rejection rule: the critical value

- For $0.05 = \Pr(1000 - \overline{X} > d)$, we have $d = 32.9$.
- As the observed sample mean $\bar{x} = 963 \in (-\infty, 967.1)$, we **reject H_0**.
  - The deviation from 1000 is large enough.
  - The evidence is strong enough.

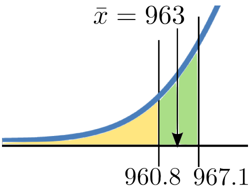


Rejection rule: the critical value

- In this example, 967.1 is the critical values for rejection.
  - If the sample mean is more extreme than (in this case, below) the critical value, we reject H_0.
  - Otherwise, we do not reject H_0.
- There is a strong evidence supporting H_a: $\mu < 1000$.
- Concluding statement:
  - Because the sample mean lies in the rejection region, we **reject H_0**. With a 95% confidence level, there **is a** strong evidence showing that the average weight **is less than** 1000 g.

One-tailed tests vs. two-tailed tests

- When should we use a two-tailed test?
  - We use a two-tailed test when we are lack of the direction information.
  - E.g., we suspect that the population mean **has changed**, but we **have no idea** about whether it becomes larger or smaller.
- If we know or believe that the change is possible **only in one direction**, we may use a one-tailed test.
- Having more information (i.e., knowing the direction of change) makes rejection "easier,", i.e., easier to find a strong enough evidence.



Summary

- Distinguish the following pairs:
  - One- and two-tailed tests.
  - No evidence showing H_0 is false and having evidence showing H_0 is true.
  - Not rejecting H_0 and accepting H_0.
  - Using $=$ and using $\geq$ or $\leq$ in the null hypothesis.

Road map

- Basic ideas of hypothesis testing.
- The first example.
- **The p-value**.

The p-value

- The **p-value** is an important, meaningful, and widely-adopted tool for hypothesis testing.

Definition 1

In a hypothesis testing, for an observed value of the statistic, the p-value is the probability of observing a value that is at least as extreme as the observed value under the assumption that the null hypothesis is true.

  - Calculated based on an **observed** value of the statistic.
  - Is the **tail probability** of the observed value.
  - Assuming that the null hypothesis is true.

The p-value

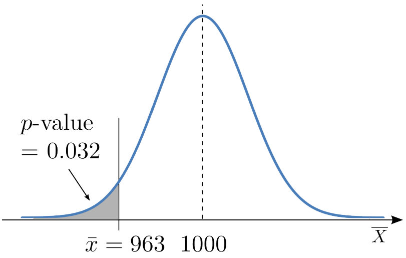
- Mathematically:
  - Suppose we test a population mean μ with a one-tailed test

$$\begin{aligned} \mathrm{H}_0 &: \mu = 1000 \\ \mathrm{H}_\mathrm{a} &: \mu < 1000. \end{aligned}$$

  - Given an observed $\bar{x}$, the p-value is defined as

$$\Pr(\overline{X} \leq \bar{x}).$$

- In the previous example, $\sigma = 200$, $n = 100$, $\alpha = 0.05$, and $\bar{x} = 963$.
  - If H_0 is true, i.e., $\mu = 1000$, we have $\Pr(\overline{X} \leq 963) = 0.032$.
  - The p-value of $\bar{x}$ is 0.032.



How to use the p-value?

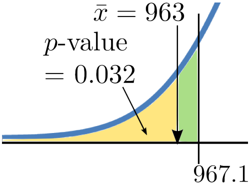
- The p-value can be used for constructing a **rejection rule**.
- For a one-tailed test:
  - If the p-value is **smaller** than α, we **reject** H_0.
  - If the p-value is greater than α, we do not reject H_0.
- In our example, the one-tailed test is

$$H_0: \mu = 1000$$
$$H_a: \mu < 1000.$$

  - We have $\alpha = 0.05$.
  - Because the p-value $0.032 < 0.05$, we reject H_0.

p-values vs. critical values

- Using the p-value is **equivalent** to using the critical values.
  - The rejection-or-not decision we make will be the same based on the two methods.



The benefit of using the p-value

- In calculating the p-value, we do not need α.
- After the p-value is calculated, we compare it with α.
- The p-value, which needs to be calculated **only once**, allows us to know whether the difference is significant under various values of α.
- In our example:

α	0.1	0.05	0.01
Rejecting H_0?	Yes ($0.032 < 0.1$)	Yes ($0.032 < 0.05$)	No ($0.032 > 0.01$)

- If we use the critical-value method, we need to calculate the critical value for three times, one for each value of α.

The benefit of using the p-value

- In many studies, researchers do not determine the significance level α **before** a test is conducted.
- They calculate the p-value and then mark **the significance** of the result with **stars**.
- One typical way of assigning stars:

p-value	Significant?	Mark
$(0, 0.01]$	Highly significant	***
$(0.01, 0.05]$	Moderately significant	**
$(0.05, 0.1]$	Slightly significant	*
$(0.1, 1)$	Insignificant	(Empty)

The benefit of using the p-value

- As an example, suppose one is testing whether people at different ages sleep for at least eight hours per day in average.
  - Age groups: $[10, 15)$, $[15, 20)$, $[20, 35)$, etc.
  - For group i, a one-tailed test is conducted. $H_a : \mu_i > 8$.
  - The result may be presented in a table:

Group	Age group	p-value
1	[10,15)	0.0002***
2	[15,20)	0.2
3	[20,25)	0.06*
4	[25,30)	0.04**
5	[30,35)	0.03**

- A smaller p-value does NOT mean **a larger deviation**!
  - We cannot conclude that $\mu_5 > \mu_4$, $\mu_1 > \mu_3$, etc.
  - There are other tests for the difference between two population means.

The p-value for two-tailed tests

- How to construct the rejection rule for a **two-tailed** test?
  - If the p-value is **smaller** than $\frac{\alpha}{2}$, we **reject** H_0.
  - If the p-value is greater than $\frac{\alpha}{2}$, we do not reject H_0.
- Consider the two-tailed test

$$H_0: \mu = 1000.$$
$$H_a: \mu \neq 1000.$$

  - We have $\alpha = 0.05$.
  - Because the p-value $0.032 > \frac{\alpha}{2} = 0.025$, we do not reject H_0.

Summary

- The p-value is the tail probability of the realized value of a statistics assuming the null hypothesis is true.
- The p-value method is an alternative way of forming the rejection rule.
  - It is equivalent to the critical-value method.
- The p-value is related to the probability for H_0 to be false.
- It does not measure the magnitude of the deviation.