Statistics and Data Analysis

Regression Analysis (2)

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Road map

- **Case study: Ticket selling**.
- Indicator variables.
- Interaction among variables.
- Endogeneity.

Ticket selling

- A **theater** made hundreds of stage performances in the past six years.
- The owner hopes that statistics and data analysis may help her improve the ticket sales.
- Key questions: What makes a show popular?
  - Popularity is defined as the **numbers of tickets sold**.
  - Potential factors: year, month, day, time, location, actors/actresses, drama type, ticket prices, etc.
- 100 performances are randomly drawn from the whole pool.
  - All were made during weekends.
  - Tickets were all publicly sold.
  - Tickets for all performances were sold through the same channels.
  - For each performance, the ticket price(s) remained the same.
- As a group of consultants, how may we help the theater?

Variables

- Six variables are obtained:

Variable	Meaning
Year	The year in which the performance was made
Time	Morning, afternoon, or evening
Capacity	The number of seats in the theater hall
AvgPrice	The average of all prices
SalesQty	The number of tickets sold
SalesDuration	Performance day − Announcement day

- Labeling and scaling:
  - Years are labeled as 1, 2, ..., and 6 (6 means the last year).
  - Capacities and sales quantities have been scaled in the same proportion.

Data

Yr.	Tm.	Cap.	A.P.	Qty	S.D.	Yr.	Tm.	Cap.	A.P.	Qty	S.D.
5	A	230	400	218	50	2	M	190	575	190	289
5	A	150	500	119	46	6	A	130	500	108	89
5	A	230	400	160	126	4	E	200	775	169	100
5	A	200	775	200	324	4	E	200	775	135	259
6	E	190	1175	178	115	5	A	310	650	251	346
6	A	190	1175	183	109	2	A	250	550	250	145
5	E	190	775	161	58	1	A	190	675	183	254
4	M	210	675	184	108	5	A	200	775	164	84
3	E	200	775	122	95	2	M	200	575	195	184
1	M	200	575	125	360	5	M	200	775	193	324
5	M	150	500	99	46	6	E	200	1175	180	74
4	A	200	775	190	262	5	A	200	775	200	82
2	E	340	550	308	78	2	M	200	575	200	35
5	A	200	775	196	170	3	E	200	775	110	89
1	E	200	575	172	359	6	M	200	1175	194	306
2	E	200	675	197	183	1	E	200	675	168	359
5	A	210	400	160	45	5	E	180	500	99	246
6	A	200	1175	200	81	4	E	200	775	194	106
1	A	200	675	192	102	3	A	250	675	181	102
3	M	200	775	198	62	3	M	200	775	148	97
6	A	200	1175	183	306	6	E	200	187.5	100	28
5	M	150	500	87	45	5	E	340	675	231	71
3	A	200	675	200	112	6	A	200	1175	146	110
5	E	200	775	158	323	1	M	200	575	140	94
1	M	200	575	128	360	4	A	200	775	195	255

Data

Yr.	Tm.	Cap.	A.P.	Qty	S.D.	Yr.	Tm.	Cap.	A.P.	Qty	S.D.
6	M	190	1175	190	107	1	A	200	675	191	355
6	A	310	1175	227	99	6	A	190	1175	190	116
4	M	200	775	200	96	3	A	200	775	149	90
6	M	200	1175	117	110	5	M	210	675	152	193
6	E	220	187.5	186	41	5	A	200	775	185	323
5	M	200	775	183	172	5	M	180	500	78	246
6	M	130	500	94	89	1	M	190	575	158	271
2	E	230	550	226	141	5	A	210	675	105	192
4	E	200	775	177	94	5	E	170	400	153	53
2	A	230	550	154	137	2	E	170	400	139	81
4	E	210	675	178	108	5	A	200	400	179	131
2	M	200	575	194	61	1	M	190	575	132	271
3	E	330	675	227	80	5	M	200	775	149	169
5	A	310	650	234	185	6	A	220	187.5	217	41
5	E	200	775	120	312	6	M	200	1175	126	311
3	A	330	675	241	81	2	E	270	550	196	177
5	E	330	675	225	255	6	M	200	1175	200	82
2	A	340	550	318	79	1	E	330	550	260	123
5	E	200	775	110	324	2	M	270	550	214	177
6	M	200	1175	200	75	5	E	200	775	84	83
4	M	200	775	199	109	2	E	200	675	198	61
2	A	340	550	294	53	6	A	200	1175	160	312
2	E	250	550	240	145	2	A	190	675	168	282
6	A	200	187.5	148	28	6	E	200	1175	137	312
1	A	230	550	219	117	5	E	360	675	227	141

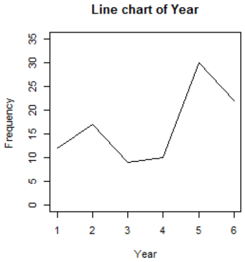
Descriptive statistics

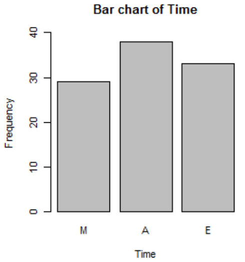
- A statistical study always starts from **descriptive statistics**.
- Some basic facts:

Year	1	2	3	4	5	6
Frequency	12	17	9	10	30	22

Time	M	A	E
Frequency	29	38	33

Variable	Min	Median	Mean	Max	St. Dev.
Capacity	130	200	216.1	360	47.78
AvgPrice	187.5	675	708.5	1175	246.99
SalesQty	78	183	176.9	318	47.04
SalesDuration	28	111	157.4	360	100.64


Line chart of Year
Frequency
0 5 10 15 20 25 30 35
1 2 3 4 5 6
Year


Bar chart of Time
Frequency
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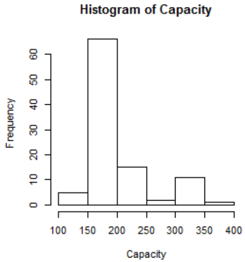
M A E
Time


Histogram of Capacity
Frequency
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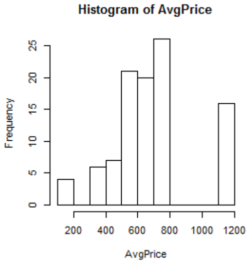
100 150 200 250 300 350 400
Capacity


Histogram of AvgPrice
Frequency
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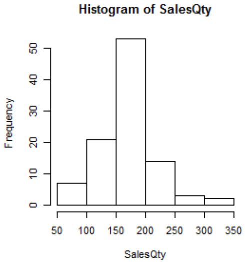
AvgPrice


Histogram of SalesQty
Frequency
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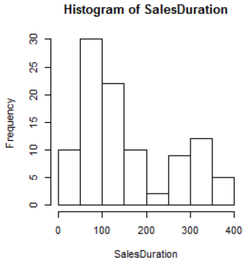
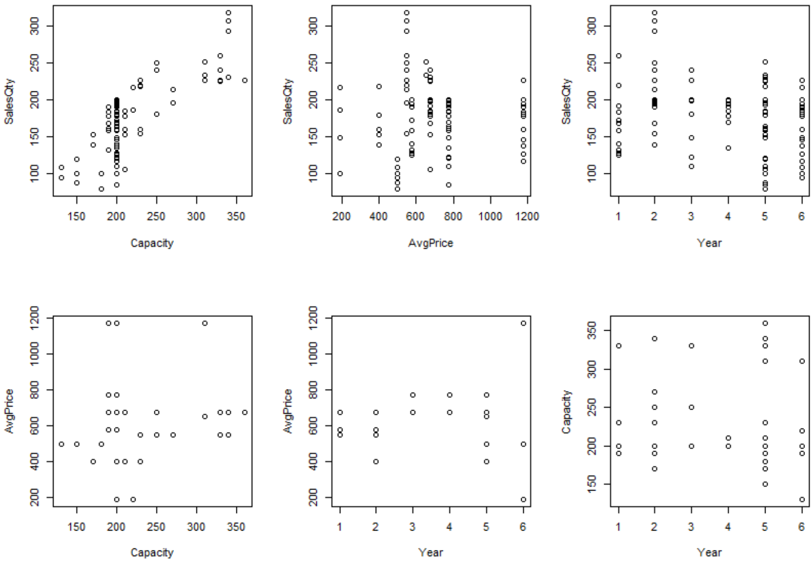
50 100 150 200 250 300 350
SalesQty


Histogram of SalesDuration
Frequency
0 5 10 15 20 25 30
0 100 200 300 400
SalesDuration

Regression

- To construct a regression model, we first consider **quantitative independent variables**.
  - Dependent variable: *SalesQty*.
  - Independent variables: *Capacity*, *AvgPrice*, *Year*.
  - Let's ignore *SalesDuration* for a while.
- Note that *Year* is a quantitative variable.
  - Indeed there are only six possible values of Year.
  - The **difference** between two values makes sense: $4 - 2$ and $5 - 3$ both mean a difference of two years.
  - The values will keep increasing.
  - If we have a variable *Month* whose possible values are 1, 2, ..., and 12, the difference between 12 and 1 is **ambiguous**: 11 months or 1 month.
- **Scatter plots** help us consider:
  - **Variable selection**: Does a variable has an impact?
  - **Transformation**: What is a variable's impact?
  - **Multicollinearity**: Are two variables highly correlated?


SalesQty
Capacity
SalesQty
AvgPrice
SalesQty
Year
AvgPrice
Capacity
AvgPrice
Year
Capacity
Year

Regression

- It seems that *Capacity*, *AvgSales*, and *Year* are all worth a try.
- Let's put them into a regression model.
- If we do this **one by one**:
  - $SalesQty = 20.79 + 0.72 Capacity$: $R^2 = 0.538$, p-value ≈ 0.
  - $SalesQty = 174.9 + 0.0028 AvgPrice$: $R^2 = 0.0002$, p-value $= 0.885$.
  - $SalesQty = 203.6 - 6.77 Year$: $R^2 = 0.063$, p-value $= 0.0115$.
- If we include them **together**:
  - The regression model is

$$SalesQty = 24.742 + 0.702 Capacity + 0.027 AvgPrice - 4.696 Year.$$

  - $R^2 = 0.57$, $R^2_{\text{adj}} = 0.556$; p-values are 0, 0.056, and 0.019, respectively.
- Do not try independent variables separately; try them together.

Adding *Time* into the model

- *Time* may also be an influential variable.
- However, it is **qualitative**.
  - More precisely, it is nominal.
  - Even if we label *Time* with numeric values, we **cannot** treat it as a quantitative variable and put it into a regression model.
- For each qualitative variable, we need to introduce several **indicator variables** to represent its values.

Road map

- Case study: Ticket selling.
- **Indicator variables**.
- Interaction among variables.
- Endogeneity.

Numeric labeling does not work

- The variable *Time* has three values.
  - Morning, afternoon, and evening.
  - Why can't we label them as 1, 2, and 3 and do regression?
- Suppose we label $(\text{morning}, \text{afternoon}, \text{evening})$ as $(1, 2, 3)$:
  - The regression model is

$$SalesQty = 164.021 + 6.313\,Time.$$

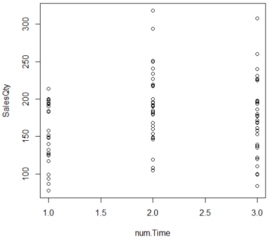
  - Why is this wrong?

Numeric labeling does not work

- **Different labeling** gives different regression results.
- We may also label (morning, afternoon, evening) as $(1, 2, 10)$ or $(3, 1, 2)$:

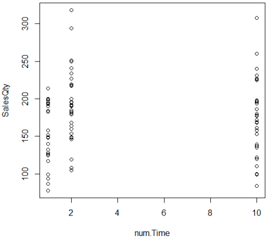
$SalesQty =$
$164.021 + 6.313\,Time$

p-value $= 0.294$



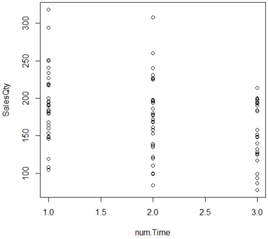
$SalesQty =$
$177.224 - 0.075\,Time$

p-value $= 0.95$



$SalesQty =$
$205.725 - 15.091\,Time$

p-value $= 0.0084$



Binary variables

- There is one exception: If a qualitative variable is **binary**, we may label the values as **0 and 1** and then treat it as quantitative.
  - Labeling values as 1 and 0, 1 and 2, or 7 and 8 is also good.
  - Labeling values as 1 and −1, 1 and 5, or 4 and 8 is bad.
- This is because a regression coefficient measures what happens to the dependent variable "when that independent variable increases **by 1**."
- When the binary variable is labeled with 0 and 1, its regression coefficient $\hat{\beta}_i$ tells us that "if the value changes from 0 to 1 (while all others remain the same), we expect the dependent variable to increase by $\hat{\beta}_i$."
- What if we have more than two values?

Indicator variables

- Consider a variable x with three values A, B, and C.
- We first choose a **reference level**, say, A.
- We then manually create two **indicator variables** x^B and x^C:

$$x^B = \begin{cases} 1 & \text{if } x = \text{B} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad x^C = \begin{cases} 1 & \text{if } x = \text{C} \\ 0 & \text{otherwise} \end{cases}$$

In other words, we have a mapping:

x	x^B	x^C
A	0	0
B	1	0
C	0	1

Indicator variables

- Lastly, we put x^B and x^C into a model to get

$$y = \hat{\beta}_0 + \cdots + \hat{\beta}^B x^B + \hat{\beta}^C x^C.$$

  - If x changes **from A to B** (and all others remain the same), we expect the dependent variable to increase by $\hat{\beta}^B$.
  - If x changes **from A to C** (and all others remain the same), we expect the dependent variable to increase by $\hat{\beta}^C$.
  - If x changes **from B to C** (and all others remain the same), we can **say nothing**.
- We use x to divide the data into three groups A, B, and C.
- We are asking, after removing the impacts from other variables, whether there is a significant difference between groups A and B (or A and C).

Indicator variables in general

- If a variable x has five values M, N, O, P, and Q.
  - We first choose a **reference level**, say, P.
  - We then manually create **four** indicator variables:

x	x^M	x^N	x^O	x^Q
M	1	0	0	0
N	0	1	0	0
O	0	0	1	0
P	0	0	0	0
Q	0	0	0	1

  - Is there a significant difference between groups P and M, P and N, P and O, and P and Q?
- In general, for a variable with k values, we need $k - 1$ indicator variables.

Adding indicator variables for *Time*

- *Time* has three values: morning, afternoon, and evening.
- Let's choose **afternoon** as the reference level.
- We need two indicator variables:

$Time$	$Time^M$	$Time^E$
morning	1	0
afternoon	0	0
evening	0	1

- Using $Time^M$ and $Time^E$ as our independent variables, we get

$$SalesQty = 191 - 30.069\,Time^M - 16.303\,Time^E,$$

  where the p-values are 0.009 and 0.138, respectively.
- If a performance is **rescheduled** from afternoon to morning, we expect the sales to decrease by 30.069.

Adding indicator variables for *Time*

- Let's include *Capacity*, *AvgPrice*, *Year*, $Time^M$, and $Time^E$:

$$\begin{aligned} SalesQty = {} & 39.28 + 0.696\,Capacity + 0.027\,AvgPrice - 5.282\,Year \\ & - 14.387\,Time^M - 21.328\,Time^E. \end{aligned}$$

	Coefficients	Standard Error	t Stat	p-value	
Intercept	39.280	19.724	1.992	0.049	**
Capacity	0.696	0.069	10.263	0.000	***
AvgPrice	0.027	0.013	2.033	0.045	**
Year	−5.282	1.931	−2.735	0.007	***
$Time^M$	−14.387	7.784	−1.848	0.068	*
$Time^E$	−21.328	7.227	−2.951	0.004	***

$R^2 = 0.608,\ R^2_{\text{adj}} = 0.587$

Summary

- When an independent variable is qualitative, we need to introduce indicator variables.
  - An indicator variable is either 0 or 1.
- If it has k possible values, we need $k-1$ indicator variables.
  - For the reference level, all indicator variables are 0.
  - For each other level, exactly one indicator variable is 1.
- We are only testing the differences between the reference level and other levels.
  - We have no idea about the difference between two non-reference levels.
  - We may change the reference level.
- As long as **one** indicator variable is significant, **all other** indicator variables for the same qualitative variable can be kept.

Interaction among variables

- Case study: Ticket selling.
- Indicator variables.
- **Interaction among variables**.
- Endogeneity.

Interaction among variables

- In a regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots \beta_p x_p,$$

  β_i measures how x_i affects y.
- Sometimes the impact of x_i on y depends on **the value of another variable** x_j.
- Consider house prices, sizes, and numbers of bedrooms.
  - When a house is big, more numbers of bedrooms may be good.
  - When a house is small, more numbers of bedrooms may be bad.
- Consider the demand of a product.
  - Demand is sensitive to price: When price goes up, demand goes down.
  - The sensitivity may be different between men and women.
- In this case, we say there is an **interaction** between x_i and x_j.

Modeling interaction

- To model the interaction between x_i and x_j, one possibility is to create a new variable $x_i x_j$, which is the **product** of the two original variables.
- In a regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{1,2} x_1 x_2 \cdots,$$

  $\beta_{1,2}$ measures the interaction between x_1 and x_2.
  - The impact of x_1 on y is $\beta_1 + \beta_{1,2} x_2$.
  - The impact of x_2 on y is $\beta_2 + \beta_{1,2} x_1$.
- A quadratic term x_i^2 in a regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_1' x_1^2 + \cdots,$$

  is a special case: The impact of x_1 on y is depends on the value of x_1.

Interaction between *Time* and *AvgPrice*

- Do *Time* and *AvgPrice* affect each other's impact?
- Let's add $Time^M \times AvgPrice$ and $Time^E \times AvgPrice$ into our model:

	Coefficients	Std. Error	t Stat	p-value	
Intercept	55.876	22.652	2.467	0.015	**
Capacity	0.676	0.068	9.950	0.000	***
Year	−6.192	1.966	−3.149	0.002	***
$Time^M$	−55.205	23.829	−2.317	0.023	**
$Time^E$	−19.105	21.81	−0.876	0.383	
AvgPrice	0.015	0.019	0.836	0.405	
$Time^M \times AvgPrice$	0.054	0.030	1.792	0.076	*
$Time^E \times AvgPrice$	−0.004	0.030	−0.136	0.892	
			$R^2 = 0.624$, $R^2_{\text{adj}} = 0.595$		

- If we want to keep $Time^E \times AvgPrice$, we must also keep $Time^M \times AvgPrice$, *AvgPrice*, $Time^M$, and $Time^E$ in our model.

Time affects *AvgPrice*'s impact

- Let's focus on *Time* and *AvgPrice*:

	Coefficients	Std. Error	t Stat	p-value	
$Time^M$	−55.205	23.829	−2.317	0.023	**
$Time^E$	−19.105	21.81	−0.876	0.383	
$AvgPrice$	0.015	0.019	0.836	0.405	
$Time^M \times AvgPrice$	0.054	0.030	1.792	0.076	*
$Time^E \times AvgPrice$	−0.004	0.030	−0.136	0.892	

- People have **different price sensitivity** for shows at different time. When the price goes up by $1, we expect:
  - The sales of an afternoon show increases by 0.015.
  - The sales of an morning show increases by $0.015 + 0.054 = 0.069$.
  - The sales of a evening show increases by $0.015 - 0.004 = 0.011$.

AvgPrice affects *Time*'s impact

- Let's focus on *Time* and *AvgPrice*:

	Coefficients	Std. Error	t Stat	p-value	
$Time^M$	−55.205	23.829	−2.317	0.023	**
$Time^E$	−19.105	21.81	−0.876	0.383	
$AvgPrice$	0.015	0.019	0.836	0.405	
$Time^M \times AvgPrice$	0.054	0.030	1.792	0.076	*
$Time^E \times AvgPrice$	−0.004	0.030	−0.136	0.892	

- If we **reschedule** an afternoon show to the morning, the impact is

$$-55.205 + 0.054 AvgPrice$$

  in expectation. If $AvgPrice = 500$, e.g., we expect the sales to decrease by $-55.205 + 0.054 \times 500 = -28.205$.
- If we reschedule an afternoon show to the evening, the expected impact is $-19.105 - 0.004 AvgPrice$.

Interaction between *Time* affects *Year*

- Do *Time* and *Year* affect each other's impact?

	Coefficients	Std. Error	t Stat	p-value	
(Intercept)	39.597	22.31	1.775	0.079	*
Capacity	0.693	0.068	10.267	0.000	***
AvgPrice	0.024	0.013	1.799	0.075	*
$Time^E$	−2.696	18.562	−0.145	0.885	
$Time^M$	−25.114	18.303	−1.372	0.173	
Year	−4.703	2.944	−1.597	0.114	
$Time^E \times Year$	−4.841	4.302	−1.125	0.263	
$Time^M \times Year$	2.898	4.166	0.695	0.489	
				$R^2 = 0.620$, $R^2_{\text{adj}} = 0.591$	

- All the five variables related to *Time* and *Year* are **insignificant**.
  - People's preference over the show time do not change from year to year.
  - The trend from year to year is the same for different show times.
- Though all the five variables are insignificant, we typically first try to remove **only the interaction terms**.

Summary

- Two variables' interaction may be modeled with a product term.
  - If its coefficient is significantly nonzero, one variable's impact depends on the other's value.
- Three rules for keeping variables:
  - Quadratic transformation: If we want to keep x^2, we must also keep x.
  - Indicator variable: If we want to keep $x^{k'}$, where $x^{k'}$ is the indicator variable for represent $x = k'$, we must also keep x^k for all $k \neq k'$.
  - Interaction: If we want to keep $x_i x_j$, we must also keep x_i and x_j.
- Therefore:
  - If we want to have $x_i x_j^{k'}$, where $x_j^{k'}$ is the indicator variable for represent $x_j = k'$, we must also keep x_j^k for all $k \neq k'$.
- It is possible to add $x_i x_j x_k$ into a regression model.

Endogeneity

- Case study: Ticket selling.
- Indicator variables.
- Interaction among variables.
- **Endogeneity**.

SalesDuration

- Consider the variable *SalesDuration*.
  - The difference between the announce day and performance day.
  - The number of days that the tickets for a show are publicly sold.
  - The longer sales duration, the more sales?
- We probably want to add *SalesDuration* into our regression model.
- This is problematic in this case:
  - Typically the theater determines its schedule for the next year at the end of each year.
  - Most performances are scheduled.
  - Ticket selling starts a few months before a series of shows are performed.
  - However, if a series turns out to be **popular**, the theater may decide to **add more shows** into this series.
  - These additional shows have much shorter *SalesDuration* and typically have high *SalesQty*.
- In short, *SalesQty* affects *SalesDuration*.

Endogeneity

- If in a regression model an independent variable is affected by the dependent variable, we say the model has the **endogeneity** problem.
  - If we add *SalesDuration* into our model, we creates endogeneity.
  - *Year*, *Time*, *Capacity*, and *AvgPrice* do not have the endogeneity problem.
  - If any of them may be modified when the theater sees a good (or bad) sales, endogeneity emerges.
- Endogeneity results in a **biased prediction**.
- In our ticket selling example, if we add *SalesDuration* into our model, we may intentionally announce shows later!

Example: promotional phone calls

- A bank lets its workers call people to invite them to deposit money.
- Many factors may affect the outcome (success or not):[1]
  - The callee' gender, age, occupation, education level, etc.
  - The caller's gender, age, experience, etc.
  - The calling day, calling time, weather at the call, etc.
- All these information from past calls are recorded.
- The **length of each call** is also recorded.
  - It is found to be highly correlated with success/failure.
  - However, it cannot be used as an independent variable.
  - Because it is **affected by the outcome**: Once one agrees to deposit money, the call gets longer to talk about more details.
- In this example, if we add call duration into our model, we may ask our workers to speak as slowly as possible.

[1] A regression model that incorporates a qualitative dependent variable will be introduced in later lectures.

Avoiding endogeneity

- To avoid endogeneity:
  - **Remove the independent variable** is endogenous.
  - **Remove those records** in which an independent variable is affected by the dependent one.
- In the ticket selling example:
  - We may remove *SalesDuration*.
  - We may remove those additional shows.
- In the promotional call example:
  - We may remove the variable of call duration.