Statistics I, Fall 2012
Homework 04

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Note. In total there are 110 points in this homework. As you may get AT MOST 100 points, you are allowed to skip 10 point. For example, if you answer nothing for Problem 1 but do all the remaining correctly, you will still get 100 points. Nevertheless, as you may make some errors, doing all problems increases your "expected" grades.

1. (10 points) Adam is the captain of a department volleyball team in NTU and Bob is a new team member. The team's regular practice is on Saturday morning, starting at 7:30. Originally, Adam believes that Bob may wake up early, late, or very late, each with probability $\frac{1}{3}$. Because Bob lives at the university dormitory, once he wakes up early, he can be on time with probability 0.9. If Bob wakes up late, he can be on time with probability 0.6. If Bob wakes up very late, he can be on time with probability 0.2.

   (a) (2 points) If Adam's belief is correct, find the probability that Bob will be late.

   (b) (2 points) Suppose Bob is late for one practice. By observing this, how would Adam adjust his belief on when Bob wakes up?

   (c) (3 points) Suppose Bob is late for two consecutive practices. By observing this, how would Adam adjust his belief on when Bob wakes up?

   (d) (3 points) Suppose after Bob is late for twice, he is on time for the third practice. By observing this, how would Adam adjust his belief on when Bob wakes up?

   Note. Once Bob is late for twice, he needs to be on time for six consecutive times so that Adam's belief on that Bob wakes up early goes above $\frac{1}{3}$. Making people distrust you is easy but reversing it can be much harder!

2. (10 points; 2 points each) Let Y be a discrete random variable with its probability distribution summarized below.

y	$\Pr(Y = y)$
0	0.15
1	0.25
2	0.35
3	0.25

   (a) Find the mean of Y, $\mathbb{E}[Y]$.

   (b) Find the variance of Y, $\text{Var}(Y)$.

   (c) Find the mean of $2 - 4Y$, $\mathbb{E}[2 - 4Y]$.

   (d) Find the variance of $2 - 4Y$, $\text{Var}(2 - 4Y)$.

   (e) Find the mean of Y^2, $\mathbb{E}\left[Y^2\right]$.

3. (12 points; 2 points each) Let Y be defined in Problem 2. Let X be a discrete random variable with its probability distribution summarized below. Suppose X and Y are independent.

x	$\Pr(X = x)$
0	0.1
1	0.4
2	0.2
3	0.2
4	0.1

(a) Find the probability that $X + Y = 3$, $\Pr(X + Y = 3)$.

(b) Find the probability that $XY = 4$, $\Pr(XY = 4)$.

(c) Find the mean of $3X + Y$, $\mathbb{E}[3X + Y]$.

(d) Find the mean of $6 + XY$, $\mathbb{E}[6 + XY]$.

(e) Find the variance of $X - 2Y$, $\text{Var}(X - 2Y)$.

(f) Find the variance of XY, $\text{Var}(XY)$.

4. (10 points) Prove that for any random variable X, $\text{Var}(X) = \mathbb{E}\big[X^2\big] - \mathbb{E}[X]^2$. Then verify this by investigating the random variable Y defined in Problem 2.

5. (8 points; 1 point each) For each of the following variables, first determine whether it is random. If yes, determine whether it is discrete or continuous.

   (a) The number of people who will graduate from NTU in 2013.

   (b) The number of boys enrolling in this course.

   (c) The amount of time you need to get to NTU from home.

   (d) The size of a randomly selected laptop used during our last lecture.

   (e) The expected value of a randomly selected laptop used during our last lecture.

   (f) The size of the laptop used by the instructor during our last lecture.

   (g) The amount of trash produced by all people in Taiwan in 2010.

   (h) The amount of trash produced by all people in Taiwan in 2015.

6. (5 points) Two candidates, Jones and Phil, are competing for a position in the government. Suppose that %45 of a population of 10000 registered voters favor Jones and %55 favor Phil. One conducts a survey by randomly selecting 100 voters and ask for their preferences. If all voters will tell the truth and select the voter that she favors, what is the probability that the survey will suggest a victory for Jones (i.e., more people select Jones)?

7. (15 points; 5 points each) A manufacturer produces a product and sell it to a retailer. The manufacturer offers a menu of three contracts. In each contract, a number of products delivered and a payment are specified. The three contracts are $(200, \$20000)$, $(300, \$29500)$, and $(400, \$38500)$. For example, if the retailer chooses the first contract, she will pay \$2000 to the manufacturer and obtain 20 units of that product. No matter how many products she obtains, the retailer will sell all of them to consumers. When she obtains 200 units, the retail price will be \$120. For 300 units, \$117. For 400 units, \$114. The product may be defective with probability $p = 2\%$. Once a consumer purchases a defective product, the retailer must pay the consumer \$200 as a penalty.

   (a) Which contract should the retailer choose to maximize her expected profit?

   (b) Suppose the retailer sets a target $t = 7.5q$, where q is the quantity purchased. She wants to maximize the probability that her profit achieves the target (e.g., if she chooses the first contract, her profit should exceed $\$7.5 \times 200 = \$1500$). Which contract should she choose?

   (c) Ignore the target set in Part (b) and consider the following. Suppose the government will charge the retailer \$2000 once there are at least ten defective products sold by her. To maximize her expected profit, which contract should the retailer choose?

8. (10 points; 5 points each) Suppose we toss a coin, whose probability of a head is p, and will stop when we see a tail. Let Z be the number of tosses we make before we stop.

   (a) Find the expected value of Z, $\mathbb{E}[Z]$.

   (b) Let $z \in \mathbb{N}$ be given and fixed, find the formula for $\Pr(Z \leq z | p)$.

9. (10 points) Suppose X and Y are two independent discrete random variables. Prove that the following statement is true: $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

   Hint. $\Pr(X = x, Y = y) = \Pr(X = x)\Pr(Y = y)$. Reading the proof of Proposition 2 on the slides for Chapter 5 helps.

   Note. This also holds for independent continuous random variables.

10. (10 points) Consider a one-dimensional random variable X (either discrete or continuous) with mean $\mu \in \mathbb{R}$. For any fixed number d, the mean squared error (MSE) of X with respect to $d \in \mathbb{R}$ is defined as $\mathbb{E}\big[(X-d)^2\big]$. Show that the MSE is minimized at $d = \mu$, i.e.,

$$\mathbb{E}\big[(X-\mu)^2\big] \leq \mathbb{E}\big[(X-d)^2\big] \quad \forall d \in \mathbb{R}.$$

   Hint. If you are on the right track, you may finish the proof in five lines. The main technique you need is taught in the first-year Calculus course. Please try to make your arguments rigorous and make your proof precise and easy to read.

   Note. This is showing that the mean minimizes the mean squared error. It can also be shown that the median minimizes the mean absolute error. Nevertheless, the proof is more complicated.

11. (10 points) You are given a coin whose probability of resulting in a head is unknown. Your job is to find the probability p. Because you really have no idea what p is, you decide to use Bayesian inference with the following steps. First, you assume $p \in \{0.1, 0.2, ..., 0.9\}$. For each of these nine possibilities, set the prior probability to $\frac{1}{9}$. Then you toss the coin several times, record the number of heads you observe, and apply Bayes' theorem. Find the posterior probability for each value of p if you observe the following:

   (a) (3 points) Six heads in ten trials.

   (b) (3 points) 65 heads in 100 trials.

   (c) (4 points) 650 heads in 1000 trials.

   Note. Now you probably have observed that, with 650 heads in 1000 trials, the value of p is most likely 0.6 or 0.7. This means that, among all nine possible values, 0.6 and 0.7 are *the most probable* probabilities that result in our observation (650 heads in 1000 trials). If you want things to be more precise, you may apply the same method on 0.61, 0.62, ..., 0.69. Would you predict the result?