Statistics I, Fall 2012
Suggested Solution for Homework 07

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1. (a) Its first moment is

$$\mathbb{E}[X] = \int_0^1 x \cdot 3x^2 dx = 3\left(\frac{1}{4}\right) x^4|_0^1 = \frac{3}{4}.$$

(b) Its third moment is

$$\mathbb{E}\big[X^3\big] = \int_0^1 x^3 \cdot 3x^2 dx = 3\left(\frac{1}{6}\right) x^6|_0^1 = \frac{1}{2}.$$

(c) First, note that

$$\mathbb{E}\big[X^k\big] = \int_0^1 x^k \cdot 3x^2 dx = 3\left(\frac{1}{k+3}\right) x^{k+3}|_0^1 = \frac{3}{k+3}.$$

The result then follows.

2. (a) The moment generating function is

$$m(t) = \mathbb{E}\big[e^{tX}\big] = \int_0^1 e^{tx} \cdot 3x^2 dx = 3\int_0^1 e^{tx} x^2 dx.$$

By integration by parts, we have

$$\int_0^1 e^{tx} x^2 dx = \frac{x^2 e^{tx}}{t}\bigg|_0^1 - \frac{2}{t}\int_0^1 x e^{tx} dx = \frac{e^t}{t} - \frac{2}{t}\int_0^1 x e^{tx} dx.$$

By another integration by parts, we have

$$\int_0^1 x e^{tx} dx = \frac{x e^{tx}}{t}\bigg|_0^1 - \frac{1}{t}\int_0^1 e^{tx} dx = \frac{e^t}{t} - \frac{e^t - 1}{t^2}.$$

It then follows that

$$m(t) = 3\left\{\frac{e^t}{t} - \frac{2}{t}\left[\frac{e^t}{t} - \frac{e^t - 1}{t^2}\right]\right\} = \frac{3}{t^3}\Big[\big(t^2 - 2t + 2\big)e^t - 2\Big].$$

(b) We have

$$\begin{aligned}
&\frac{d}{dt} m(t) = m'(t) \\
&= \left(\frac{-9}{t^4}\right)\Big[\big(t^2 - 2t + 2\big)e^t - 2\Big] + \left(\frac{3}{t^3}\right)\Big[(2t-2)e^t + \big(t^2 - 2t + 2\big)e^t\Big] \\
&= \left(\frac{-9}{t^4}\right)\Big[\big(t^2 - 2t + 2\big)e^t - 2\Big] + \frac{3e^t}{t} \\
&= \frac{3}{t^4}\Big[\big(t^3 - 3t^2 + 6t - 6\big)e^t + 6\Big].
\end{aligned}$$

(c) We have

$$\begin{aligned}
\lim_{t\to 0} m'(t) &= \lim_{t\to 0} \frac{3\Big[\big(t^3 - 3t^2 + 6t - 6\big)e^t + 6\Big]}{t^4} \\
&= \lim_{t\to 0} \frac{3\Big[\big(3t^2 - 6t + 6\big)e^t + \big(t^3 - 3t^2 + 6t - 6\big)e^t\Big]}{4t^3} \\
&= \lim_{t\to 0} \frac{3t^3 e^t}{4t^3} = \lim_{t\to 0} \frac{3}{4} e^t = \frac{3}{4},
\end{aligned}$$

which is the first moment as we calculated in Problem 1.

3. The moment generating function of $X \sim Uni(a,b)$ is

$$m(t) = \mathbb{E}\big[e^{tX}\big] = \int_a^b e^{tx} \cdot \frac{1}{b-a} dx = \frac{1}{b-a}\left(\frac{1}{t}\right) e^{tx}|_a^b = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

4. (a) The moment generating function of $\overline{X}$ is

$$\mathbb{E}\big[e^{t\overline{X}}\big] = \mathbb{E}\big[e^{t(X_1+\cdots X_n)/n}\big] = \mathbb{E}\big[e^{(t/n)X_1+\cdots(t/n)X_n}\big] = \mathbb{E}\big[e^{(t/n)X_1}\big]\cdots\mathbb{E}\big[e^{(t/n)X_n}\big],$$

where the last equality is due to the independence among X_is. Now, because $X_i \sim \text{ND}(\mu, \sigma)$ for all i, we have

$$\mathbb{E}\big[e^{t\overline{X}}\big] = \left\{\exp\left[\mu(t/n) + \frac{\sigma^2}{2}(t/n)^2\right]\right\}^n = \left[\exp\left(\mu t + \frac{\sigma^2/n}{2}t^2\right)\right],$$

which is the moment generating function of a normal random variable with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$. It then follows that $\overline{X} \sim \text{ND}(\mu, \frac{\sigma}{\sqrt{n}})$.

(b) To see this, recall that $\overline{X}$ itself is a normal random variable. For any normal random variable, its standardization results in a standard normal random variable (according to the proposition of linear functions of normal random variables we proved in the lecture). Therefore, as we know

$$\overline{X} \sim \text{ND}\left(\mu, \frac{\sigma}{\sqrt{n}}\right),$$

that proposition implies the desired result.

5. The moment generating function of $X_1 + X_2$ is

$$\mathbb{E}\big[e^{t(X_1+X_2)}\big] = \mathbb{E}\big[e^{tX_1} \cdot e^{tX_2}\big] = \mathbb{E}\big[e^{tX_1}\big] \cdot \big[e^{tX_2}\big],$$

where the last equality is due to the independence between X_1 and X_2. As $X_i \sim \text{Bi}(n_i, p)$, $i = 1, 2$, its moment generating function is $[pe^t + (1-p)]^{n_i}$. Therefore, we have

$$\mathbb{E}\big[e^{t(X_1+X_2)}\big] = [pe^t + (1-p)]^{n_1} \cdot [pe^t + (1-p)]^{n_2} = [pe^t + (1-p)]^{(n_1+n_2)},$$

which is the moment generating function of a binomial random variable with $n_1 + n_2$ trials and probability p. It then follows that $X_1 + X_2 \sim \text{Bi}(n_1 + n_2, p)$.

6. Let $\overline{X}$ be the sample mean and Z be a standard normal random variable.

(a) According to Problem 4a, the distribution of the sample mean is $\overline{X} \sim \text{ND}(120, \frac{40}{\sqrt{n}}$.

(b) When $n = 16$, $\overline{X} \sim \text{ND}(120, 10)$. The desired probability is

$$1 - \Pr\left(\overline{X} \in [114, 126]\right) = 1 - \Pr\left(Z \in [-0.6, 0.6]\right) \approx 1 - (0.726 - 0.274) = 0.549.$$

(c) When $n = 100$, $\overline{X} \sim \text{ND}(120, 4)$. The desired probability is

$$1 - \Pr\left(\overline{X} \in [114, 126]\right) = 1 - \Pr\left(Z \in [-1.5, 1.5]\right) \approx 1 - (0.933 - 0.067) = 0.134.$$

(d) Given any sample size n, we want

$$1 - \Pr\left(\overline{X} \in [114, 126]\right) = 1 - \Pr\left(Z \in \left[\frac{-6}{40/\sqrt{n}}, \frac{6}{40/\sqrt{n}}\right]\right) \le 0.01,$$

i.e., $\Pr(Z > \frac{6}{40/\sqrt{n}}) < 0.005$. This requires

$$\frac{6}{40/\sqrt{n}} \ge 2.576 \quad \Leftrightarrow \quad \sqrt{n} \ge 2.576 \times \frac{40}{6} \approx 17.17 \quad \Leftrightarrow \quad n \ge 294.88.$$

The smallest sample size allowed is thus 295.