1. Consider the reseller introduced in the lecture videos. Suppose that there is no sales agent, and the reseller obtains a demand signal s_R according to the rule introduced in the lecture videos.

   (a) Under the assumption that $\gamma = \frac{1}{2}$, find $\Pr(\theta = \theta_L | s_R = G)$.

   (b) Under the assumption that $\gamma = \frac{1}{2}$, find $\Pr(\theta = \theta_H | s_R = G)$.

   (c) Under the assumption that $\gamma = \frac{1}{2}$, find $\mathbb{E}[\theta | s_R = G]$.

   (d) Given any $\gamma \in (0, 1)$, find $\mathbb{E}[\theta | s_R = G]$.

2. Consider the notation N_{jk} in the lecture video, which is

$$N_{jk} = \mathbb{E}[\theta | s_A = j, s_R = k].$$

(a) Under the assumption $\gamma = \frac{1}{2}$, show that

$$N_{FG} = \frac{\theta_H \lambda_A \lambda_R + \theta_L (1 - \lambda_A)(1 - \lambda_R)}{\lambda_A \lambda_R + (1 - \lambda_A)(1 - \lambda_R)}.$$

(b) Find N_{FB}.

(c) Given any $\gamma \in (0, 1)$, find N_{FB}. May you show that

$$N_{FB} > \gamma \theta_L + (1 - \gamma) \theta_H$$

if and only if $\lambda_R < \lambda_A$?

(d) Consider the notation $\bar{P}_{jk} = \Pr(s_A = j | s_R = k)$. Show that

$$\bar{P}_{FG} = \lambda_A \lambda_R + (1 - \lambda_A)(1 - \lambda_R).$$

3. Recall the reseller's contract design problem

$$\begin{aligned}
\mathcal{R}_k \equiv \max_{\substack{\alpha_{\mathrm{F}} \text{ urs.}, \beta_{\mathrm{F}} \geq 0, \\ \alpha_{\mathrm{U}} \text{ urs.}, \beta_{\mathrm{U}} \geq 0}} \quad & \sum_{j \in \{F,U\}} \bar{P}_{jk} \Big[u - \alpha_j + (v - \beta_j) N_{jk}^2 \beta_j \Big] \\
\text{s.t.} \quad & \alpha_{\mathrm{F}} + \frac{1}{2} \beta_{\mathrm{F}}^2 N_{Fk}^2 \geq 0 \\
& \alpha_{\mathrm{U}} + \frac{1}{2} \beta_{\mathrm{U}}^2 N_{Uk}^2 \geq 0 \\
& \alpha_{\mathrm{F}} + \frac{1}{2} \beta_{\mathrm{F}}^2 N_{Fk}^2 \geq \alpha_{\mathrm{U}} + \frac{1}{2} \beta_{\mathrm{U}}^2 N_{Fk}^2 \\
& \alpha_{\mathrm{U}} + \frac{1}{2} \beta_{\mathrm{U}}^2 N_{Uk}^2 \geq \alpha_{\mathrm{F}} + \frac{1}{2} \beta_{\mathrm{F}}^2 N_{Uk}^2 .
\end{aligned}$$

Solve the problem to find the reseller's optimal menu.

4. **Note.** The next four problems are connected.

Consider a salesperson who may privately exert sales effort a. The cost of exerting effort a is $\frac{1}{2}a^2$. The sales quantity is $x \in \{0, 1\}$, which follows a Bernoulli distribution

$$\Pr(x = 1|a) = a = 1 - \Pr(x = 0|a).$$

In other words, the higher effort exerted by the salesperson, the higher probability the good is sold. Let the production cost be 0 and retail price be 1. Let the salesperson be the one that earns the sales revenue. Suppose that the salesperson's objective is to maximize his expected profit, find his optimal effort level and the associated profit.

5. Following from the previous problem, suppose that the salesperson is hired by a retailer. The retailer offers the salesperson a contract $\{\alpha, \beta\}$, where $\alpha \in \mathbb{R}$ is the fixed salary and $\beta \in [0, 1]$ is the sales bonus. In other words, the salesperson's earning is $\alpha + \beta x$. Suppose that both players' objectives are to maximize each of their own expected profit.

   (a) Suppose that the salesperson has selected the contract (α, β), find his optimal effort and the associated profit as functions of α and β.

   (b) Find the retailer's optimal contract.

   (c) Does the salesperson earn any information rent? Briefly explain why.

6. Following from the previous problem, suppose that the salesperson is protected by *limited liability*: α is restricted by $\alpha \geq 0$ during the contracting stage. All other settings remain the same.

   (a) Suppose that the salesperson has selected the contract (α, β), find his optimal effort and the associated profit as functions of α and β.

   (b) Find the retailer's optimal contract.

   (c) Does the salesperson earn any information rent? Briefly explain why.

7. Following from the previous problem, suppose that the salesperson is *risk-averse*: Instead of maximize his expected profit, he maximizes his expected profit minus *one half* of the *variance* of his earning. All other settings remain the same.

   (a) Formulate the sales agent's utility function.

   (b) Suppose that the salesperson has selected the contract (α, β), find his optimal effort and the associated profit as functions of α and β.

   (c) Find the retailer's optimal contract.

   (d) Does the salesperson earn any information rent? Briefly explain why.

8. Consider a salesperson who may privately observe the market condition θ. The sales quantity is $x \in \{0, 1\}$, which follows a Bernoulli distribution

$$\Pr(x = 1|\theta) = \theta = 1 - \Pr(x = 0|\theta).$$

In other words, the better market condition, the higher probability the good is sold. θ follows another Bernoulli distribution

$$\Pr(\theta = \theta_L) = \frac{1}{2} = 1 - \Pr(\theta = \theta_H),$$

where $0 < \theta_L < \theta_H < 1$. Let the production cost be 0 and retail price be 1. Suppose that the salesperson is hired by a retailer. For each type of salesperson, the retailer offers a contract $\{\alpha, \beta\}$, where $\alpha \in \mathbb{R}$ is the fixed salary and $\beta \in [0, 1]$ is the sales bonus. In other words, the salesperson's earning is $\alpha + \beta x$. Suppose that both players' objectives are to maximize each of their own expected profit.

(a) Explain why the retailer should offer a menu of two contracts.

(b) Formulate the retailer's contract design problem.

(c) Find the retailer's optimal menu of contracts.