Information Economics

The Continuous-type Screening Model

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Road map

- **Introduction**.
- Preliminaries.
- Optimal contracts.
- Implications.

Screening

- Recall our monopoly pricing screening problem:
  - There are two kinds of consumers:
    - $\theta \in \{\theta_\text{L}, \theta_\text{H}\}$ where $\theta_\text{L} < \theta_\text{H}$, is the consumer's private information.
    - The seller believes that $\Pr(\theta = \theta_\text{L}) = \beta = 1 - \Pr(\theta = \theta_\text{H})$.
  - When obtaining q units by paying t, a type-θ consumer's utility is

$$u(q, t, \theta) = \theta v(q) - t.$$

    - $v(q)$ is strictly increasing and strictly concave. $v(0) = 0$.
  - The unit production cost of the seller is $c < \theta_\text{L}$.
  - By selling q units and receiving t, the seller earns $t - cq$.
  - How would you price your product to maximize your expected profit?
- Because we assume that there are two kinds of consumers, this is a **two-type** screening model.

Two-type screening

- The two-type screening problem can be formulated:

$$\begin{aligned} \max_{q_{\rm H}, t_{\rm H}, q_{\rm L}, t_{\rm L}} \quad & \beta\Big[t_{\rm L} - cq_{\rm L}\Big] + (1-\beta)\Big[t_{\rm H} - cq_{\rm H}\Big] \\ \text{s.t.} \quad & \theta_{\rm H} v(q_{\rm H}) - t_{\rm H} \geq \theta_{\rm H} v(q_{\rm L}) - t_{\rm L} \\ & \theta_{\rm L} v(q_{\rm L}) - t_{\rm L} \geq \theta_{\rm L} v(q_{\rm H}) - t_{\rm H} \\ & \theta_{\rm H} v(q_{\rm H}) - t_{\rm H} \geq 0 \\ & \theta_{\rm L} v(q_{\rm L}) - t_{\rm L} \geq 0. \end{aligned}$$

  - The first two are the incentive-compatible (truth-telling) constraints.
  - The last two are the individual-rationality (participation) constraints.
- If $\frac{\theta_{\rm H} - \theta_{\rm L}}{\theta_{\rm H}} < \beta$, the optimal menu $\{(q_{\rm L}^*, t_{\rm L}^*), (q_{\rm H}^*, t_{\rm H}^*)\}$ satisfies

$$\theta_{\rm H} v'(q_{\rm H}^*) = c \quad \text{and} \quad \theta_{\rm L} v'(q_{\rm L}^*) = c\left[\frac{1}{1 - \left(\frac{1-\beta}{\beta}\,\frac{\theta_{\rm H} - \theta_{\rm L}}{\theta_{\rm L}}\right)}\right].$$

- May we generalize this problem to n **types**?

n-type screening

- Let $\theta \in \{\theta_1, \theta_2, ..., \theta_n\}$, where $\theta_1 < \theta_2 < \cdots < \theta_n$ and $\Pr(\theta = \theta_i) = \beta_i$.
  - Of course we have $\beta_i > 0$ and $\sum_{i=1}^n \beta_i = 1$.
- The n-type screening problem can be formulated:

$$
\begin{aligned}
\max_{\{q_i, t_i\}} \quad & \sum_{i=1}^n \beta_i (t_i - cq_i) \\
\text{s.t.} \quad & \theta_i v(q_i) - t_i \geq \theta_i v(q_j) - t_j \quad \forall i = 1, ..., n, j = 1, ..., n \\
& \theta_i v(q_i) - t_i \geq 0 \quad \forall i = 1, ..., n.
\end{aligned}
$$

  - The first set is the set of IC constraints.
  - The second set is the set of IR constraints.
- How to find the optimal menu?

n-type screening

- The n-type screening problem can be reduced to:

$$\begin{aligned} \max_{\{q_i, t_i\}} \quad & \sum_{i=1}^{n} \beta_i (t_i - cq_i) \\ \text{s.t.} \quad & \theta_i v(q_i) - t_i \geq \theta_i v(q_{i-1}) - t_{i-1} \quad \forall i = 2, ..., n \\ & \theta_1 v(q_1) - t_1 \geq 0. \end{aligned}$$

  - Only local downward IC constraints (LDIC) are necessary.
  - Only the IR constraint for the lowest type is necessary.
  - Monotonicity, efficiency at top, and no rent at bottom still hold.
- May we generalize this problem to **infinitely many types** on a **continuum**?

Continuous-type screening

- Let $\theta \in S = [\theta_0, \theta_1]$, where $\theta_0 < \theta_1$, with f and F as the pdf and cdf.
- The continuous-type screening problem can be formulated:

$$\max_{\{q(\theta), t(\theta)\}} \quad \int_{\theta_0}^{\theta_1} \Big[t(\theta) - cq(\theta) \Big] f(\theta) d\theta$$

$$\text{s.t.} \quad \theta v(q(\theta)) - t(\theta) \geq \theta v(q(\hat{\theta})) - t(\hat{\theta}) \quad \forall \theta \in S, \hat{\theta} \in S$$

$$\theta v(q(\theta)) - t(\theta) \geq 0 \quad \forall \theta \in S.$$

  - The first set is the set of IC constraints.
  - The second set is the set of IR constraints.
- How to find the optimal menu?

Road map

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- **Preliminaries**.
- Optimal contracts.
- Implications.

Preliminaries

- Before we try to solve for the optimal menu, we need to get some mathematical tools.
  - Hazard (failure) rates.
  - Integration by parts.
  - Envelope theorem.

Failure (hazard) rates

- Consider a bulb whose life is $X \geq 0$. Let $X \sim f, F$.
  - $F(t) = \Pr(X \leq t)$ is the probability for the bulb to fail by time t.
  - $F(t+\epsilon) - F(t)$ is the probability for the bulb to fail within $[t, t+\epsilon]$.
  - $f(t) = \frac{d}{dt}F(t) = \lim_{\epsilon \to 0}[F(t+\epsilon) - F(t)]$ is the probability density for the bulb to fail at time t.
- The **failure (hazard) rate** of the bulb $h(t)$ is the likelihood for the bulb to fail at time t, given that the bulb has not failed by time t:

$$h(t) = \lim_{\epsilon \to 0} \Pr\left(X \in [t, t+\epsilon] \middle| X \geq t\right) = \lim_{\epsilon \to 0} \frac{\Pr(X \in [t, t+\epsilon], X \geq t)}{\Pr(X \geq t)}$$

$$= \lim_{\epsilon \to 0} \frac{\Pr(X \in [t, t+\epsilon])}{1 - F(t)} = \frac{f(t)}{1 - F(t)}.$$

Failure (hazard) rates

- Some examples:
  - If $X \sim \text{Uni}(0, 1)$, we have $f(x) = 1$, $F(x) = x$, and thus $h(x) = \frac{1}{1-x}$. The hazard rate is increasing.
  - If $X \sim \text{Exp}(\lambda)$, we have $f(x) = \lambda e^{-\lambda x}$, $F(x) = 1 - e^{-\lambda x}$, and thus $h(x) = \lambda$. The hazard rate is constant.
- In general, for a random variable with pdf $f(\cdot)$ and cdf $F(\cdot)$, its failure rate is $h(\cdot) = \frac{f(\cdot)}{1-F(\cdot)}$.
- For our private type θ, we impose the following assumption:

Assumption 1 (Increasing failure rate (IFR))

The failure rate of θ is (weakly) increasing: Let $H(\theta) = \frac{1-F(\theta)}{f(\theta)}$, then $H(\theta)$ is (weakly) decreasing in θ.

  - This is true for most of the well-known distributions (uniform, exponential, normal, gamma, beta, etc.).

Integration by parts

- Let $u(x)$ and $v(x)$ be two functions of x defined over $[a, b]$. We have

$$\frac{d}{dx}\Big[u(x)v(x)\Big] = \Big[u(x)v(x)\Big]' = u(x)v'(x) + v(x)u'(x).$$

- Integrating both sides with respect to x:

$$\int_a^b \frac{d}{dx}\Big[u(x)v(x)\Big]dx = \int_a^b u(x)v'(x)dx + \int_a^b v(x)u'(x)dx$$

$$\Leftrightarrow \quad \int_a^b u(x)v'(x)dx = \Big[u(x)v(x)\Big]\Big|_a^b - \int_a^b v(x)u'(x)dx.$$

- The (abbreviated) formula of **integration by parts**:

$$\int udv = uv - \int vdu.$$

Integration by parts: examples

- Find $\int_0^1 xe^x dx$:

$$\int_0^1 \underbrace{x}_{u}\underbrace{e^x dx}_{dv} = \underbrace{x}_{u}\underbrace{e^x}_{v}\Big|_0^1 - \int_0^1 \underbrace{e^x}_{v}\underbrace{dx}_{du} = e - (e-1) = 1.$$

- Find $\int_0^1 x^2 e^x dx$:

$$\int_0^1 \underbrace{x^2}_{u}\underbrace{e^x dx}_{dv} = \underbrace{x^2}_{u}\underbrace{e^x}_{v}\Big|_0^1 - \int_0^1 \underbrace{e^x}_{v}\underbrace{2x dx}_{du} = e - 2 \times 1 = e - 2.$$

A parameter's impact on the objective value

- Consider a function $f(x, \theta)$ and an optimization problem

$$z^*(\theta) = \max_x \ f(x, \theta).$$

  We will interpret x as the decision variable and θ as the parameter. $z^*(\theta)$ is the maximum attainable objective value given θ.
- Let $x^*(\theta) \in \operatorname{argmax}_x f(x, \theta)$ be an optimal solution. Then we have

$$z^*(\theta) = f(x^*(\theta), \theta).$$

- Question: What is $\frac{d}{d\theta} z^*(\theta)$, the impact of θ on the objective value?
  - One application: the impact of a parameter on the equilibrium utility.

Envelope theorem

- An example: Let $f(x,\theta) = \theta - (x-\theta)^2$. Given θ fixed, we have $x^*(\theta) = \theta$ and $z^*(\theta) = \theta - (\theta - \theta)^2 = \theta$. Therefore, $\frac{d}{d\theta} z^*(\theta) = 1$.
- To find $\frac{d}{d\theta} z^*(\theta)$ in general:
  - Find $x^*(\theta)$, plug in $x^*(\theta)$, and then take the derivative.
  - May we "reverse the order?"
- With the **envelope theorem**, we can:
  - Find $x^*(\theta)$, take the derivative (typically easier), and then plug in $x^*(\theta)$.

Proposition 1 (Envelope theorem)

Given $f(x,\theta)$, let $x^*(\theta) \in \operatorname{argmax}_x f(x,\theta)$ and $z^*(\theta) = f(x^*(\theta),\theta)$. Then we have

$$\frac{d}{d\theta} z^*(\theta) = \left. \frac{\partial f(x,\theta)}{\partial \theta} \right|_{x = x^*(\theta)}.$$

Envelope theorem

Proof. We have

$$\begin{aligned}\frac{d}{d\theta}z^*(\theta) &= \frac{d}{d\theta}f(x^*(\theta),\theta) \\ &= \left(\frac{\partial f(x,\theta)}{\partial x}\cdot\frac{\partial x^*(\theta)}{d\theta}+\frac{\partial f(x,\theta)}{\partial\theta}\right)\Bigg|_{x=x^*(\theta)} \\ &= \frac{\partial f(x,\theta)}{\partial x}\Bigg|_{x=x^*(\theta)}\cdot\frac{\partial x^*(\theta)}{d\theta}+\frac{\partial f(x,\theta)}{\partial\theta}\Bigg|_{x=x^*(\theta)} \\ &= 0\cdot\frac{\partial x^*(\theta)}{d\theta}+\frac{\partial f(x,\theta)}{\partial\theta}\Bigg|_{x=x^*(\theta)} = \frac{\partial f(x,\theta)}{\partial\theta}\Bigg|_{x=x^*(\theta)}.\end{aligned}$$

The second equation follows the total differential formula. The second last equation comes from the fact that $x^*(\theta)$ satisfies the first-order condition of $f(x,\theta)$. □

Envelope theorem: examples

- Consider $f(x,\theta) = \theta - (x-\theta)^2$:

$$\frac{d}{d\theta} z^*(\theta) = \frac{\partial f(x,\theta)}{\partial \theta}\bigg|_{x=x^*(\theta)} = \Big[1 + 2(x-\theta)\Big]\Big|_{x=\theta} = 1.$$

- Consider $f(x,\theta) = -\frac{1}{3}x^3 + \theta x$ over $x \in [0,\infty)$ for some $\theta > 0$.
  - Without the envelope theorem, we do:

$$x^*(\theta) = \sqrt{\theta}, \quad z^*(\theta) = f(x^*(\theta),\theta) = \frac{2}{3}\sqrt{\theta^3}, \quad \text{and then } \frac{d}{d\theta} z^*(\theta) = \sqrt{\theta}.$$

  - With the envelope theorem, we do:

$$x^*(\theta) = \sqrt{\theta}, \quad \frac{\partial f(x,\theta)}{\partial \theta} = x, \quad \text{and then } \frac{d}{d\theta} z^*(\theta) = x|_{x=\sqrt{\theta}} = \sqrt{\theta}.$$

  Note that $\frac{\partial f(x,\theta)}{\partial x}|_{x=\sqrt{\theta}} = (-x^2+\theta)|_{x=\sqrt{\theta}} = 0$.

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Solving the contract design problem

- Now we are going to solve

$$\max_{\{q(\theta),t(\theta)\}} \quad \int_{\theta_0}^{\theta_1} \Big[t(\theta) - cq(\theta)\Big] f(\theta) d\theta$$

$$\text{s.t.} \quad \theta v(q(\theta)) - t(\theta) \geq \theta v(q(\hat{\theta})) - t(\hat{\theta}) \quad \forall \theta \in S, \hat{\theta} \in S$$

$$\theta v(q(\theta)) - t(\theta) \geq 0 \quad \forall \theta \in S,$$

  where $S = [\theta_0, \theta_1]$ is the set of types. Note that there are infinitely many variables and constraints.
- Strategy:
  - Monotonicity: Higher types consume more.
  - IR: Show that only the IR constraint for the lowest type is necessary.
  - IC: Show that only local IC constraints are necessary.
  - Using binding constraints to get an unconstrained problem.
  - Pointwise optimization.

Step 1: Monotonicity

- Consider two types θ and $\hat{\theta}$. Let $\theta > \hat{\theta}$. We have the two IC constraints between them:

$$\theta v(q(\theta)) - t(\theta) \geq \theta v(q(\hat{\theta})) - t(\hat{\theta})$$

and

$$\hat{\theta} v(q(\hat{\theta})) - t(\hat{\theta}) \geq \hat{\theta} v(q(\theta)) - t(\theta).$$

- Adding them together, we obtain

$$\begin{aligned}
& \theta v(q(\theta)) + \hat{\theta} v(q(\hat{\theta})) \geq \theta v(q(\hat{\theta})) + \hat{\theta} v(q(\theta)) \\
\Leftrightarrow\ & (\theta - \hat{\theta}) v(q(\theta)) \geq (\theta - \hat{\theta}) v(q(\hat{\theta})) \\
\Leftrightarrow\ & v(q(\theta)) \geq v(q(\hat{\theta})) \\
\Leftrightarrow\ & q(\theta) \geq q(\hat{\theta}).
\end{aligned}$$

- Therefore, $\theta > \hat{\theta}$ implies $q(\theta) \geq q(\hat{\theta})$. It can be shown to be $q'(\theta) \geq 0$.

Step 2: only one IR constraint is not redundant

- Consider a type $\theta > \theta_0$. We have

$$\theta v(q(\theta)) - t(\theta) \geq \theta v(q(\theta_0)) - t(\theta_0) \geq \theta_0 v(q(\theta_0)) - t(\theta_0) \geq 0.$$

- Therefore, only $\theta_0 v(q(\theta_0)) - t(\theta_0) \geq 0$ is necessary.
- This is the **lowest-type** IR constraint.

Step 3: local IC + monotonicity = global IC

- The reduced program:

$$\max_{\{q(\theta),t(\theta)\}} \quad \int_{\theta_0}^{\theta_1} \Big[t(\theta) - cq(\theta)\Big] f(\theta) d\theta$$

$$\text{s.t.} \quad \theta v(q(\theta)) - t(\theta) \geq \theta v(q(\hat{\theta})) - t(\hat{\theta}) \quad \forall \theta \in S, \hat{\theta} \in S$$

$$\theta_0 v(q(\theta_0)) - t(\theta_0) \geq 0.$$

- We now want to reduce the set of global IC constraints.
- Let's first rewrite them:

$$\theta \in \operatorname*{argmax}_{\hat{\theta} \in S} \Big\{ \theta v(q(\hat{\theta})) - t(\hat{\theta}) \Big\} \quad \forall \theta \in S.$$

  - It should be optimal for a consumer to **report his true type**.
- Our target: local IC + monotonicity = global IC.

Step 3: local IC + monotonicity = global IC

- Let $W(\theta, \hat{\theta}) = \theta v(q(\hat{\theta})) - t(\hat{\theta})$. This is a type-$\theta$ consumer's utility by misreporting his type as $\hat{\theta}$.
- **Global IC**: $\theta \in \underset{\hat{\theta}}{\operatorname{argmax}}\ W(\theta, \hat{\theta})$.
- If θ is globally optimal, it must also be locally optimal. Therefore, it must satisfy the FOC:

$$
\begin{aligned}
& \frac{\partial}{\partial \hat{\theta}} W(\theta, \hat{\theta}) \bigg|_{\hat{\theta}=\theta} = 0 \\
\Leftrightarrow & \left[\theta v'(q(\hat{\theta})) q'(\hat{\theta}) - t'(\hat{\theta}) \right] \bigg|_{\hat{\theta}=\theta} = 0 \\
\Leftrightarrow & \ \theta v'(q(\theta)) q'(\theta) - t'(\theta) = 0.
\end{aligned}
$$

  The last equality is the set of **local IC** constraints.
- **Monotonicity**: $q'(\theta) \geq 0$.

Step 3: local IC + monotonicity = global IC

- To show that local IC + monotonicity = global IC, we need to show:
  - Local IC + monotonicity $\Leftarrow$ global IC.
  - Local IC + monotonicity $\Rightarrow$ global IC.
- The first one is obvious: (1) Global IC implies local IC by definition. (2) Global IC implies monotonicity has been shown in Step 1.
- If the second one is false, there exists θ such that $W(\theta, \hat{\theta}) - W(\theta, \theta) > 0$ for some $\hat{\theta}$. Without loss of generality, let $\hat{\theta} > \theta$. We have

$$W(\theta, \hat{\theta}) - W(\theta, \theta) = \int_{\theta}^{\hat{\theta}} \frac{\partial W(\theta, x)}{\partial x} dx$$

$$= \int_{\theta}^{\hat{\theta}} \Big[\theta v'(q(x))q'(x) - t'(x)\Big] dx \leq \int_{\theta}^{\hat{\theta}} \Big[x v'(q(x))q'(x) - t'(x)\Big] dx = 0,$$

where the inequality relies on $q'(x) \geq 0$ and the last equality relies on local IC. This contradicts with $W(\theta, \hat{\theta}) - W(\theta, \theta) > 0$.

Step 4: ignoring monotonicity

- The reduced program:

$$\begin{aligned}
\max_{\{q(\theta),t(\theta)\}} \quad & \int_{\theta_0}^{\theta_1} \Big[t(\theta) - cq(\theta)\Big] f(\theta) d\theta \\
\text{s.t.} \quad & \theta v'(q(\theta))q'(\theta) - t'(\theta) = 0 \quad \forall \theta \in S \\
& q'(\theta) \geq 0 \quad \forall \theta \in S \\
& \theta_0 v(q(\theta_0)) - t(\theta_0) \geq 0.
\end{aligned}$$

- Let's **ignore the monotonicity constraints** for a while. We will verify that the optimal solution of the relaxed program satisfies the monotonicity constraints.

Step 5: finding the unconstrained program

- The reduced program:

$$\max_{\{q(\theta),t(\theta)\}} \quad \int_{\theta_0}^{\theta_1} \Big[t(\theta) - cq(\theta)\Big] f(\theta) d\theta$$

$$\text{s.t.} \quad \theta v'(q(\theta))q'(\theta) - t'(\theta) = 0 \quad \forall \theta \in S$$

$$\theta_0 v(q(\theta_0)) - t(\theta_0) \geq 0.$$

- Let $W(\theta) = W(\theta, \theta) = \max_{\hat{\theta} \in S} W(\theta, \hat{\theta})$ be the type-θ consumer's equilibrium utility under truth-telling. By the **envelope theorem**:

$$W'(\theta) = \frac{\partial}{\partial \theta} W(\theta, \hat{\theta}) \Big|_{\hat{\theta}=\theta} = \frac{\partial}{\partial \theta} \Big[\theta v(q(\hat{\theta})) - t(\hat{\theta})\Big] \Big|_{\hat{\theta}=\theta}$$

$$= v(q(\hat{\theta}))|_{\hat{\theta}=\theta} = v(q(\theta)) \geq 0.$$

- One may prove this by using **local IC** instead of the envelope theorem.

Step 5: finding the unconstrained program

- With $W'(\theta) = v(q(\theta))$, we have

$$W(\theta) = \int_{\theta_0}^{\theta} v(q(x))dx + W(\theta_0),$$

where $W(\theta_0) = \theta_0 v(q(\theta_0)) - t(\theta_0) \geq 0$ is the type-θ_0 consumer's equilibrium utility.

- Because $v(q(\theta)) \geq 0$ implies $W(\theta) \geq W(\theta_0)$ for all $\theta \geq \theta_0$, we have $W(\theta_0) = 0$ at any optimal solution (otherwise we should increase $t(\theta_0)$).
- Now the only IR constraint is satisfied (as a binding constraint).
- Now we have $W(\theta) = \int_{\theta_0}^{\theta} v(q(x))dx$. Local IC implies

$$t(\theta) = \theta v(q(\theta)) - W(\theta) = \theta v(q(\theta)) - \int_{\theta_0}^{\theta} v(q(x))dx.$$

- Let's plug in $t(\theta)$ into the objective function.

Step 6: solving the unconstrained program

- The reduced program:

$$\begin{aligned}&\max_{\{q(\theta)\}} \int_{\theta_0}^{\theta_1} \Big[t(\theta) - cq(\theta)\Big] f(\theta) d\theta \\ = &\max_{\{q(\theta)\}} \int_{\theta_0}^{\theta_1} \left[\theta v(q(\theta)) - \int_{\theta_0}^{\theta} v(q(x))dx - cq(\theta)\right] f(\theta) d\theta.\end{aligned}$$

- How to simplify the objective function?

Step 6: solving the unconstrained program

- With **integration by parts**, we have

$$\int_{\theta_0}^{\theta_1} \underbrace{\int_{\theta_0}^{\theta} v(q(x))dx}_{u} \underbrace{f(\theta)d\theta}_{dv} = \underbrace{\int_{\theta_0}^{\theta} v(q(x))dx}_{u} \underbrace{F(\theta)}_{v} \Big|_{\theta_0}^{\theta_1} - \int_{\theta_0}^{\theta_1} \underbrace{F(\theta)}_{v} \underbrace{v(q(\theta))d\theta}_{du}$$

$$= \int_{\theta_0}^{\theta_1} v(q(\theta))d\theta - \int_{\theta_0}^{\theta_1} F(\theta)v(q(\theta))d\theta = \int_{\theta_0}^{\theta_1} \Big[1 - F(\theta)\Big] v(q(\theta))d\theta.$$

- The reduced program:

$$\max_{\{q(\theta)\}} \int_{\theta_0}^{\theta_1} \left[\theta v(q(\theta)) - \int_{\theta_0}^{\theta} v(q(x))dx - cq(\theta)\right] f(\theta)d\theta$$

$$= \max_{\{q(\theta)\}} \int_{\theta_0}^{\theta_1} \left[\theta v(q(\theta)) - \frac{1 - F(\theta)}{f(\theta)} v(q(\theta)) - cq(\theta)\right] f(\theta)d\theta.$$

Step 6: solving the unconstrained program

- To solve

$$\max_{\{q(\theta)\}} \int_{\theta_0}^{\theta_1} \left[\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)v(q(\theta)) - cq(\theta)\right] f(\theta)d\theta,$$

  we do **pointwise optimization**.
- For each θ, maximize $(\theta - \frac{1-F(\theta)}{f(\theta)})v(q(\theta)) - cq(\theta)$ with respect to $q(\theta)$.
- For each θ, the optimal $q^*(\theta)$ satisfies the FOC[1]

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)v'(q^*(\theta)) = c.$$

- $t^*(\theta)$ can be found as $t^*(\theta) = \theta v(q^*(\theta)) - \int_{\theta_0}^{\theta} v(q^*(x))dx$.

[1] If for some θ the equation cannot be satisfied, e.g., when $\theta - \frac{1-F(\theta)}{f(\theta)} < 0$, we have $q^*(\theta) = 0$.

Step 7: final checks

- Our solution $q^*(\theta)$ satisfies $(\theta - \frac{1-F(\theta)}{f(\theta)})v'(q^*(\theta)) = c$.
- We need to verify that $q^*(\theta)$ satisfies **monotonicity** and **local IC**.
- Monotonicity:
  - By assumption, $\frac{1-F(\theta)}{f(\theta)}$ decreases in θ.
  - Therefore, $\theta - \frac{1-F(\theta)}{f(\theta)}$ increases in θ.
  - Therefore, $v'(q^*(\theta))$ decreases in θ.
  - As $v'(\cdot)$ is decreasing, we have $q^*(\theta)$ increases in θ.
- Local IC:
  - Our optimal contracts satisfy $t(\theta) = \theta v(q(\theta)) - \int_{\theta_0}^{\theta} v(q(x))dx$.
  - Differentiate both sides with respect to θ:

$$t'(\theta) = \theta v'(q(\theta))q'(\theta) + v(q(\theta)) - v(q(\theta)) = \theta v'(q(\theta))q'(\theta).$$

    This is exactly local IC.

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Monotonicity and no rent at bottom

- Recall the main characteristics of our two-type screening model:
  - Monotonicity.
  - Efficiency at top.
  - No rent at bottom.
- **Monotonicity** has been verified.
- **No rent at bottom** is a result of the binding IR constraint for θ_0.
  - To see it from the optimal contracts, note that

$$t(\theta_0) = \theta_0 v(q(\theta_0)) - \int_{\theta_0}^{\theta_0} v(q(x))dx = \theta_0 v(q(\theta_0)).$$

  - As $W(\theta) = \theta v(q(\theta)) - t(\theta)$, $W(\theta_0) = 0$.
  - All higher types earn positive utilities (**information rents**).
  - No rent at bottom becomes no rent **only** at bottom.

Efficiency at top

- To illustrate **efficiency at top**, note that the first-best quantity $q^{\text{FB}}(\theta)$ and the second-best quantity $q^*(\theta)$ satisfy

$$\theta v'(q^{\text{FB}}(\theta)) = c \quad \text{and} \quad \left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) v'(q^*(\theta)) = c,$$

  respectively.
  - As $\frac{1-F(\theta)}{f(\theta)} > 0$ for all $\theta < \theta_1$, we have $\theta > \theta - \frac{1-F(\theta)}{f(\theta)}$ for all $\theta < \theta_1$.
  - This implies that $v'(q^{\text{FB}}(\theta)) < v'(q^*(\theta))$.
  - As $v'(\cdot)$ is decreasing, we have $q^{\text{FB}}(\theta) > q^*(\theta)$ for all $\theta < \theta_1$.
  - Only for θ_1 we have $\frac{1-F(\theta_1)}{f(\theta_1)} = 0$ and thus $q^{\text{FB}}(\theta_1) = q^*(\theta_1)$
- Except for θ_1, there is a **downward distortion on quantity**.
  - Efficiency at top becomes efficiency **only** at top.
  - This is to prevent a high type from **mimicking a low type**.
  - The principal cuts down **information rents** while sacrificing **efficiency**.

Summary

- A screening model with an infinitely many types of agents on a continuum is introduced.
- Implications from the two-type model are valid and extended:
  - Monotonicity throughout the continuum.
  - Efficiency only at top.
  - No rent only at bottom.
- We also learn/review some useful concepts/techniques:
  - Hazard (failure) rates.
  - Integration by parts.
  - Envelope theorem.
- A continuous-type model can be useful:
  - More general than the two-type model.
  - Less tedious than the n-type model.