Linear Algebra and its Applications, Spring 2013
Suggested Solution for Final Exam

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1. (a) We have

$$A = \begin{bmatrix} 1 & 2 & 5 & 1 & 0 & 4 \\ 1 & 2 & 4 & 0 & -3 & 4 \\ 3 & 6 & 14 & 4 & -3 & 13 \\ 2 & 4 & 10 & 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 & 1 & 0 & 4 \\ 0 & 0 & -1 & -1 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} = LU.$$

   (b) Because the column space is the entire $\mathbb{R}^4$, the projection of any vector in $\mathbb{R}^4$ onto the column space is the same vector. Therefore, the projection of d is $(3, 2, 4, 5)$.

2. (a) We have $\det A = \det(Q\Lambda Q^T) = \det Q \det \Lambda \det Q^T = \det \Lambda \det(Q^T Q) = \det \Lambda \det I = \det \Lambda$, which is exactly the product of A's eigenvalues.

   (b)
   - For B, first note that $\det B = 24 \det B'$, where B' is a 4×4 matrix satisfying $B'_{ij} = 1$ if $i \neq j$ or 0 if $i = j$. For B', note that three of its eigenvalues are -1, which then implies that the last eigenvalue is 3 (because the sum of eigenvalues are the trace, which is 0). Then we know that $\det B' = -3$, the product of the eigenvalues. Collectively, $\det B = -72$.
   - For C, which is a generalization of B, we follow the same way. First, $\det C = n! \det C'$, where C' is an $n \times n$ matrix satisfying $C'_{ij} = 1$ if $i \neq j$ or 0 if $i = j$. For the eigenvalues of C', $n - 1$ are -1 and one is $n - 1$. Therefore, $\det C' = (-1)^{n-1}(n - 1)$ and thus $\det C = (n!)(-1)^{n-1}(n - 1)$.

3. (a) $u_1 = Au_0 = (0.2, 0.3, 0.5)$.

   (b) $u_2 = Au_1 = (0.29, 0.36, 0.35)$.

   (c) Because the sum of each column in A is 1 and all entries in A are no greater than 1, A is Markov. This implies that the process is neutrally stable.

   (d) As eigenvectors corresponding to the eigenvalue 1 are $\{(5k, 6k, 5k)\}_{k\in\mathbb{R}}$, $u_\infty = (\frac{5}{16}, \frac{3}{8}, \frac{5}{16})$.

   (e) The problem is not meaningful. Two points are given to you for free.

   (f) The problem is not meaningful. Two points are given to you for free.

4. (a) False. A should be symmetric.

   (b) False. When A's eigenvalues are all real, A may be asymmetric.

   (c) False. The A in the next problem provides an example.

   (d) True. A is unitary implies that A is normal, which then implies that A can have n independent eigenvectors of A.

   (e) True. If A's eigenvalues are all positive, A is positive definite. Therefore, A can be decomposed into $A = R^T R$, where R has independent columns. As R has n columns, it must have n pivots.

5. We have

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{3}{\sqrt{14}} & \frac{-2}{\sqrt{21}} \\ \frac{-2}{\sqrt{6}} & \frac{2}{\sqrt{14}} & \frac{1}{\sqrt{21}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{14}} & \frac{4}{\sqrt{21}} \end{bmatrix}^T.$$

6. (a) Let x_1 and x_2 be eigenvector's of A associated with λ_1 and λ_2, respectively. Then we have $\lambda_1 x_1^H x_2 = x_1^H A^H x_2 = x_1^H A x_2 = \lambda_2 x_1^H x_2$, where the second equality comes from the fact that A is positive semidefinite (and thus symmetric). Because $\lambda_1 \neq \lambda_2$, we have $x_1^H x_2 = 0$, i.e., x_1 and x_2 are orthogonal.

(b) An lower bound is 0, because all eigenvalues must be nonnegative for A to be positive semidefinite.

(c) Suppose μ is an eigenvalue of $A - kI$, we have $(A - kI)x = \mu x$, i.e., $Ax = (\mu + k)x$. This implies that $\mu + k$ is an eigenvalue of A. Conversely, if λ is an eigenvalue of A, we have $Ax = \lambda x$, which implies that $(A - kI)x = (\lambda - k)x$. We thus know that μ is an eigenvalue of $A - kI$ if and only if $\mu + k$ is an eigenvalue of A. Therefore, if the eigenvalues of A are λ_1, λ_2, ..., λ_n, the eigenvalues of $A - kI$ must be $\lambda_1 - k$, $\lambda_2 - k$, ..., $\lambda_n - k$.

(d) The statement is false, because a stationary point x must satisfy $Ax = kx$, which cannot be satisfied if k is not an eigenvalue of A.

(e) A point (x, k) is a stationary point if and only if $Ax = kx$, which requires k to be an eigenvalue of A. For (x, k) to be a local minimum, we need $A - kI$, the Hessian matrix of $f(x, k)$, to be positive semidefinite. This requires that all eigenvalues of $A - kI$ to be nonnegative, and this happens if and only if k is the smallest eigenvalue of A. Therefore, there is a unique local minimum $(x_{\text{min}}, \lambda_{\text{min}}$, where $\lambda_{\text{min}} = \min_{i=1,\ldots,n}\{\lambda_i$ and x_{min} is the associated eigenvector.

7. (a) An upper bound is $\binom{5}{2} = 10$.

(b) An upper bound is $\binom{6}{3} = 20$.

(c) There may be at most ten distinct bases. However, because columns 1 and 4 are dependent, x_1 and x_4 cannot form a basis. Similarly, x_2 and x_5 cannot form a basis. Therefore, there are eight distinct bases.

(d) The statement is false. For example, if $n = m = 2$ and the two equality constraints are $2x_1 + x_2 = 6$ and $x_1 + 2x_2 = 6$, there is only one basic solution $(2, 2)$, which is also a basic feasible solution.

8. (a) The Hessian matrix is

$$A_2 = \begin{bmatrix} 2r & -2 \\ -2 & 2r \end{bmatrix}.$$

(b) For $\pi(p)$ to be strictly convex, we need A to be positive definite. This requires that $2r \geq 0$ and $\det A_2 = 4r^2 - 4 \geq 0$, which together imply $r > 1$. Therefore, $\pi(p)$ is strictly convex if and only if $r > 1$.

(c) The Hessian matrix is

$$A_3 = \begin{bmatrix} 2r & -2 & -2 \\ -2 & 2r & -2 \\ -2 & -2 & 2r \end{bmatrix}.$$

(d) For $\pi(p)$ to be strictly convex, we need A to be positive definite. To find a condition for positive definiteness, we calculate the eigenvalues of A_3. Clearly if we can make all columns identical, there will be two eigenvalues associated with those identical columns. Therefore, two eigenvalues will be $2r + 2$, which is always positive. The last eigenvalue can be found by $6r - 2(2r + 2) = 2r - 4$ (the sum of eigenvalues is the trace) or recognizing that we may have an eigenvalue making the three columns sum to 0. It then follows that all we need is $2r - 4 > 0$, i.e., $r > 2$.

(e) Similar to the case of $n = 3$, for a general n, there will be $n - 1$ eigenvalues being $2r + 2$. The last eigenvalue will be $2nr - (n - 1)(2r + 2) = 2r - 2(n - 1)$. The necessary and sufficient condition we need is $r > n - 1$.