Linear Algebra and its Applications, Spring 2013
Homework 13

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Note 1. This homework is due ***8:30 am, December 17, 2013***. Please submit a hard copy into the homework box outside the TAs' lab.

Note 2. "Problem sets" should be found in the textbook (the fourth edition).

1. (10 points; 5 points each) Find the extreme points of the following sets.
   (a) $\{1, 2, 3, 4\}$.
   (b) $\{x \in \mathbb{R}^2 | 0 \leq x_1 \leq 1, 0 \leq x_2 \leq 3 - 2x_1\} \cup \{x \in \mathbb{R}^2 | 1 \leq x_1 \leq 2, 0 \leq 2x_2 \leq 3 - x_1\}$.

2. (20 points) Let $S = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ and $C = \{x \in \mathbb{R}^n | Ax \leq b, -Ax \leq -b, -Ix \leq 0\}$ be two polyhedra, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and I is the $n \times n$ identity matrix.
   (a) (5 points) Show that the two polyhedra are identical.
   (b) (15 points) Show that a point is a basic feasible solution of S if and only if it is a basic feasible solution of C.
   Hint. Apply Definitions 4 and 5 in the slides.
   (c) (0 point) Convince yourself that you have shown that a point is a basic feasible solution of a standard form LP if and only if it is an extreme point of the feasible region.

3. (50 points) Consider the following LP

$$\begin{aligned} \min \quad & -3x_1 - 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0. \end{aligned}$$

   (a) (5 points) Graphically solve the LP and find an optimal solution.
   (b) (5 points) Find the standard form by adding x_3, x_4, and x_5 into the three inequalities. What else do you need to do?
   (c) (10 points) For the standard form LP in Part (b), find all the basic solutions. Among them, who are basic feasible solutions?
   (d) (10 points) Let an initial basis be $B_0 = \{3, 4, 5\}$. Do one simplex iteration to find a next basis.
   (e) (10 points) Suppose your friend Eren has done several iterations and stopped at $B_1 = \{1, 2, 4\}$. Show that B_1 is not an optimal basis WITHOUT referring to your answer in Part (a).
   (f) (10 points) Suppose your friend Mikasa has done several iterations and stopped at $B_2 = \{1, 2, 5\}$. Show that B_2 is an optimal basis WITHOUT referring to your answer in Part (a).

4. (20 points) Consider a pair of LPs below with B as an optimal basis of (P):

$$(P) \quad \begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad \text{and} \qquad (D) \quad \begin{aligned} \max \quad & y^T b \\ \text{s.t.} \quad & y^T A \leq c^T \\ & y \in \mathbb{R}^m. \end{aligned}$$

   (a) (10 points) Show that $\bar{y}^T = c_B^T A_B^{-1}$ is a feasible solution to (D).
   Hint. What is the reduced cost with respect to B?
   (b) (5 points) Show that $y^T b \leq c^T \bar{x}$ if $\bar{x}$ is optimal to (P) and y is feasible to (D).
   (c) (5 points) Show that $\bar{y}$ defined in Part (a) is optimal to (D) by showing that $\bar{y}^T b = c^T \bar{x}$.
   Hint. Given an optimal basis B, what is $c^T \bar{x}$? Let $\bar{x}$ be $(\bar{x}_B, \bar{x}_N)$ and express $\bar{x}$ by A_B!