# Operations Research, Spring 2013
# Final Exam

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**Note 1.** In total there are 110 points for this exam. If you get more than 100 points, your official score for this exam will only be 100.

**Note 2.** You do not need to return these problem sheets. Write down all your answers on the answer sheets provided to you.

1. (10 points) Consider the following linear program

$$\begin{aligned} \max \quad & 2x_1 + x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & x_1 - 2x_2 \geq 2 \\ & x_3 \leq 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

   Use the simplex method to find an optimal solution, show it is infeasible, or show it is unbounded.

2. (15 points; 3 points each) Consider the following mathematical program

$$\begin{aligned} \min \quad & f(y_1, y_2, y_3, y_4) \\ \text{s.t.} \quad & g(y_1, y_2, y_3, y_4) \leq 0 \\ & 0 \leq y_j \leq 1 \quad \forall j = 1, ..., 4. \end{aligned}$$

   Let $y = (y_1, y_2, y_3, y_4)$, $f(y) = f(y_1, y_2, y_3, y_4)$, and $g(y) = g(y_1, y_2, y_3, y_4)$. Answer the following five questions separately. For Parts (c), (d), and (e), you MUST show that your answer is optimal. If you fail to do so, you will only get partial credits even if your answer is an optimal solution.

   (a) Write down the condition for this program to be a convex program.

   (b) Write down the Lagrangian relaxation by relaxing only the constraint $g(y) \leq 0$.

   (c) Let $f(y) = 3y_1 + 7y_2 + 2y_3 + 6y_4$ and $g(y) = -4y_1 - 3y_2 - 4y_3 - 2y_4 + 7$. Find an optimal solution.

   **Hint.** Is this similar to the continuous knapsack problem?

   (d) Let $f(y) = \sqrt{7y_1 + 8y_2 + 12y_3} + \sqrt{13y_1 + 10y_2 + 8y_3} - (2y_1 + 3y_2 + 4y_3)$ and $g(y) = y_4$. Find an optimal solution.

   (e) Let $f(y) = y_1^2 + 2y_1 + 2y_2^2 - y_2 + e^{2y_3} + \sqrt{y_4 + 3}$ and $g(y) = 0$. Find an optimal solution.

3. (10 points) Answer the following questions by assuming that all assumptions for the EOQ model are valid.

   (a) (3 points) Suppose for a certain product, the annual demand quantity is 1000 unit, ordering cost is \$20 per order, annual holding cost is \$1 per unit. What is the optimal order quantity?

   (b) (3 points) Suppose for another product, the annual demand quantity is 2000 unit, ordering cost is \$5 per order, annual holding cost is \$2 per unit. What is the optimal order quantity?

   (c) (4 points) Continue from Parts (a) and (b). Suppose the company store all inventory in its warehouse, which can store at most 250 units of these two products in total. Can the company follow the EOQ rule to order the two products described in Parts (a) and (b)? Why or why not?

4. (20 points) An airline company is deciding to set up several offices in $n$ cities. If there are offices in both cities $i$ and $j$, the company can choose to operate a flight between these two cities and earn $R_{ij}$ as the annual revenue. The annual cost of operating such a flight is $C_{ij}$. The annual cost of operating an office in city $i$ is $F_i$. The company wants to find an optimal plan for operating offices and flights to maximize its annual profit.

   (a) (5 points) Formulate an integer program that can be used to find the company's optimal plan.

   (b) (10 points) Suppose now people not only want to take direct flights but also are willing to take indirect flights. More precisely, the company earns revenues from a pair of cities $i$ and $j$ according to the following rule: (1) If there is a direct flight, the company earns $R_{ij}$. (2) If there is no direct flight but there is at least one indirect flight which stops at exactly one city except $i$ and $j$, the company earns $\alpha R_{ij}$ for some known $\alpha \in (0, 1)$. (3) If none of the above is true, the company earns nothing.[1] Under this new setting, formulate an integer program that can be used to find the company's optimal plan.

   (c) (5 points) Ignore Part (b) and focus on your integer program in Part (a). Suppose you relax all binary variables to be fractional between 0 and 1. If you solve the resulting linear program and find an optimal solution, will it always be an integer solution? Why or why not?

5. (10 points) One seller is selling one single item to $n$ bidders through auction. The auction rule is the following. First, each bidder submits a bid, whose value cannot be observed by other bidders, to the buyer. Then the buyer compares all the bids and sell the item to the bidder who placed the highest bid. The winning bidder then pays the seller the second highest bid. As an example, suppose 4 bidders place bids \$10, \$12, \$15, and \$5, bidder 3 buys the item by paying \$12. Each bidder has his valuation about the item, which is unobservable by anyone else, and places his bid to maximize his expected utility. If he wins the product, his utility is his valuation minus the amount he pays. If he does not win, his utility is zero. Let $x_i$ be bidder $i$'s valuation and $b_i$ be the bid placed by bidder $i$. Explain why that all bidders bid at their valuations, i.e., choosing $b_i = x_i$, is a Nash equilibrium. For simplicity, you may assume that all bidders have different valuations and thus ignore the possibility for multiple players to win at the same time.

   **Hint.** To explain this, think in the following way: If all other bidders bid at their valuations, is there any reason for me to deviate and bid higher or lower than my valuation?

6. (15 points; 5 points each) Consider the following two-player zero-sum game

| | E | F |
|---|---|---|
| A | 2 | $-2$ |
| B | 1 | $-1$ |
| C | 0 | 0 |
| D | $-1$ | 2 |

   where the numbers are player 1's payoff. Let player 1 be the one whose action space is {A, B, C, D} and player 2 be the one whose action space is {E, F}. Player 1 acts to maximize her expected payoff and player 2 acts to minimize player 1's expected payoff.

   (a) Show that there is no saddle point.

   (b) As there is no saddle point, formulate a linear program whose optimal solution is player 1's equilibrium mixed strategy.

   (c) Solve player 1's problem by finding an optimal solution to the linear program you formulated in Part (b).

[1] As an example, suppose among cities 1, 2, 3, and 4 there are three direct flights between 1 and 2, 2 and 3, and 3 and 4. The company then earns $R_{12} + R_{23} + R_{34} + \alpha(R_{13} + R_{24})$. Note that the company cannot earn anything from the pair of cities 1 and 4 because no person can travel between these two cities with no more than one stop.

7. (25 points; 5 points each) In a supply chain, there is a manufacturer and a retailer. The manufacturer produces a product at a unit production cost $c$ and then sell the product to the retailer at a unit wholesale price $w$. While $c$ is given and fixed, $w$ can be chosen by the manufacturer to maximize her profit. The retailer faces a market in which the retail price has been fixed at $r$. The demand in the coming selling season is a random variable $D$. The retailer chooses an order quantity $q$ to maximize his expected profit. In this supply chain, first the manufacturer chooses the wholesale price $w$ and then the retailer chooses the order quantity $q$. After the selling season starts, the retailer will have no more chance to procure the product.

   (a) Suppose $D$ is uniformly distributed between 0 and $b$. Formulate the retailer's problem as a newsvendor problem. Then find the retailer's optimal order quantity $q^*(w)$ as a function of $w$ (and other parameters).

   (b) Continue from Part (a). Formulate the manufacturer's problem with the retailer's best response $q^*(w)$. Then find the manufacturer's optimal wholesale price $w^*$.

   (c) Continue from Parts (a) and (b). What is the retailer's equilibrium order quantity $q^* = q^*(w^*)$? What is the retailer's expected profit in equilibrium? What is the manufacturer's profit in equilibrium?

   (d) Continue from Parts (a), (b), and (c). Suppose the manufacturer now faces consumers directly. There is no retailer, no wholesale price, and the manufacturer chooses the production quantity $Q$ by herself. Formulate the manufacturer's problem as a newsvendor problem and find the optimal order quantity $Q^*$. Show that $Q^* > q^*$.

   (e) Suppose $D$ is a general continuous random variable with probability density function $f$ and cumulative distribution function $F$. Let the manufacturer's optimal production quantity be $Q^*$ when she sells to consumers directly and the retailer's equilibrium order quantity be $q^*$ when the manufacturer sells through the retailer. Prove that $Q^* > q^*$.

8. (5 points) In your opinion, what is the most difficult part of this course? If you are the instructor of this course, what will you do to make students learn more?

   **Note.** As you may expect, you will get 5 points as long as you write down anything reasonable. Therefore, work on this problem seriously only if you have time.