Operations Research, Spring 2013
Homework 03

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1. (30 points; 3 points each) Answer the following True/False questions for a standard form LP with n variables and m constraints. Provide brief explanations.

   (a) The maximum number of distinct basic solutions is $\binom{n}{m}$.

   (b) The maximum number of distinct bases is $\binom{n}{m}$.

   (c) The number of distinct basic feasible solution is exactly $\binom{n}{m}$.

   (d) The number of distinct bases is exactly $\binom{n}{m}$.

   (e) A basic feasible solution has exactly m adjacent basic feasible solutions.

   (f) A basic feasible solution has exactly $m(n-m)$ adjacent basic feasible solutions.

   (g) A basic feasible solution has at most $m(n-m)$ adjacent basic feasible solutions.

   (h) If a basic feasible solution has no better adjacent basic feasible solution, it is optimal.

   (i) If an LP is finitely optimal, those optimal solutions must all be basic solutions.

   (j) If an LP is finitely optimal, those optimal solutions must all be basic feasible solutions.

2. (Modified from Problem 4.1.3; 10 points) Convert the following LP to its standard form:

$$\begin{aligned} \min \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 \geq 3 \\ & x_1 + x_2 \leq 4 \\ & 2x_1 - x_2 = 3 \\ & x1, x2 \geq 0. \end{aligned}$$

3. (Modified from Problem 4.4.1; 10 points) Consider the following LP

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 100 \\ & x_1 + x_2 \leq 80 \\ & x_1 \leq 40 \\ & x_1, x_2 \geq 0. \end{aligned} \tag{1}$$

   (a) (5 points) Find the standard form.

   (b) (5 points) For the standard form LP in Part (b), find all the basic solutions. Among them, who are basic feasible solutions?

4. (Modified from Problem 4.4.2; 10 points) Consider the following LP (Example 2 in Chapter 3)

$$\begin{aligned} \min \quad & 50x_1 + 100x_2 \\ \text{s.t.} \quad & 7x_1 + 2x_2 \geq 28 \\ & 2x_1 + 12x_2 \geq 24 \\ & x_1, x_2 \geq 0. \end{aligned}$$

   show how the basic feasible solutions to the LP in standard form correspond to the extreme points of the feasible region. Certainly you need to draw the feasible region of the original problem.

5. (10 points) Consider the following LP:

$$\begin{aligned} \min \quad & -x_1 + x_2 \\ \text{s.t.} \quad & -2x_1 + x_2 \geq 6 \\ & x_1 - x_2 = -6 \\ & x_1 \leq 0, x_2 \geq 0. \end{aligned} \tag{2}$$

(a) (5 points) Convert the following LP to its standard form:

(b) (5 points) For the LP in (2), show how the basic feasible solutions to its standard form correspond to the extreme points of the feasible region of the original LP in (2).

Note. Keep in mind that the *original* LP is the one having x_1 *nonpositive*.

6. (30 points) Consider the LP we discussed in class

$$\begin{array}{llllll} \min & -x_1 & & & & \\ \text{s.t.} & 2x_1 & - & x_2 & \leq & 4 \\ & 2x_1 & + & x_2 & \leq & 8 \\ & & & x_2 & \leq & 3 \\ & x_i \geq 0 & \forall\, i = 1, 2. & & & \end{array}$$

and its standard form

$$\begin{array}{llllllllllll} \min & -x_1 & & & & & & & & & & \\ \text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\ & 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\ & & & x_2 & & & & & + & x_5 & = & 3 \\ & x_i \geq 0 & \forall\, i = 1, ..., 5. & & & & & & & & & \end{array}$$

The feasible region of the original LP is depicted in Figure 1. As a convention, a variable with a superscript (e.g, x^3) means a vector/point and one with a subscript (e.g., x_3) means a single variable (e.g., x_3 is the first slack variable).

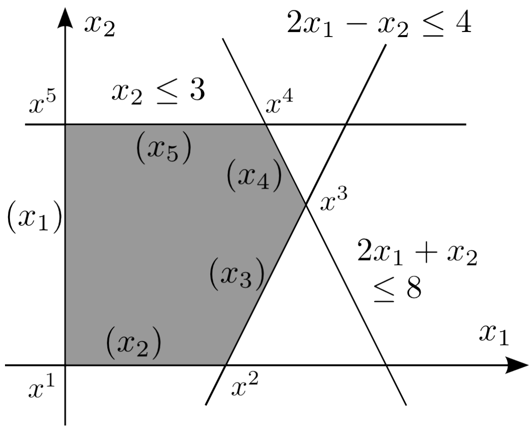

Figure 1: Feasible region for the LP in Problem 6.

(a) (5 points) Note that on each constraint in Figure 1, there is an associated variable contained in a pair of parentheses. For example, for constraint $2x_1 + x_2 \leq 8$, the associated variable is x_4. Show that the associated variable is a slack variable for a functional constraint but an original variable for a nonnegativity constraint.

Note. Providing a simple mapping between constraints and variables is enough.

(b) (0 points) The meaning of the associated variable is that "the associated variable remains 0 along the boundary of this constraint." For example, on the line $2x_1 + x_2 = 8$, which is the boundary of the constraint $2x_1 + x_2 \leq 8$, x_4 is always 0. Convince yourself that this holds for all constraints.

(c) (5 points) If a point lies on one line, we know the associate variable is 0 for that point. For example, the point x^3 lies on the line $2x_1 + x_2 \leq 8$, so x_4 must be 0 for the corresponding solution to the standard form. To see this, first note that $x^3 = (3, 2)$ (by solving a two by two linear system) and $2 \times 3 + 2 + x_4 = 8$, which means the slack variable x_4 is 0.[1] Using the same method to show that for the point x^4, the variable x_4 and x_5 are both 0.

(d) (10 points) Given each extreme point in Figure 1, there is a corresponding basic feasible solution of the standard form. With the observation in Part (b), now you realize one thing: Given an extreme point, you can determine which two variables in the corresponding basic feasible solution is nonbasic according to the constraints the extreme point lies on. For example, for the extreme point x^3, variables x_4 and x_5 are nonbasic. Find all the basic feasible solutions, draw connection between basic feasible solutions and extreme points, and then show that the above statement is true.

(e) (5 points) OK, now given an extreme point, you know who are basic and who are nonbasic in its corresponding basic feasible solution. It is then possible to verify that adjacent basic feasible solutions indeed lie on the same edge of the feasible region. Show that x^4 and x^5 are adjacent but x^4 and x^2 are not.

(f) (5 points) Suppose I am at point x^3 and I want to move to x^4. What are my entering and leaving variables? Suppose I am at point x^5 and I want to move to x^4. What are my entering and leaving variables?

[1]This also means that there is no gap between the two sides of the constraint $2x_1 + x_2 \leq 8$.