Operations Research, Spring 2013
Suggested Solution for Homework 04

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1. (a) The standard form is

$$\begin{aligned} z^* = \ \max \quad & 2x_1 - x_2 + x_3 \\ \text{s.t.} \quad & 3x_1 + x_2 + x_3 + x_4 = 60 \\ & x_1 - x_2 + 2x_3 + x_5 = 10 \\ & x_1 + x_2 - x_3 + x_6 = 20 \\ & x_i \geq 0 \quad \forall\, i = 1, ..., 6, \end{aligned}$$

where x_4, x_5, and x_6 are the slack variables.

(b) We have

$$A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix},$$
$$c_B = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \text{ and } \quad c_N = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}.$$

It then follows that

$$A_B^{-1}A_N = \begin{bmatrix} 3 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}, \quad A_B^{-1}b = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix},$$
$$c_B A_B^{-1} b = 0, \text{ and } \quad \bar{c} = c_B A_B^{-1} A_N - c_N = \begin{bmatrix} -2 & 1 & -1 \end{bmatrix}.$$

(c) The initial tableau is

$$\begin{array}{cccccc|c} -2 & 1 & -1 & 0 & 0 & 0 & 0 \\ \hline 3 & 1 & 1 & 1 & 0 & 0 & 60 \\ 1 & -1 & 2 & 0 & 1 & 0 & 10 \\ 1 & 1 & -1 & 0 & 0 & 1 & 20 \end{array}$$

$A_B^{-1}A_N$ is in the constraint rows of nonbasic columns (1, 2, and 3), $A_B^{-1}b$ is the RHS column, $c_B A_B^{-1} b$ is in the right-top corner, and $\bar{c}$ is in the objective row of nonbasic columns.

(d) We have

$$A_B = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 60 \\ 10 \\ 20 \end{bmatrix},$$
$$c_B = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}, \text{ and } \quad c_N = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}.$$

It then follows that

$$A_B^{-1}A_N = \begin{bmatrix} -3 & 4 & -5 \\ 1 & -1 & 2 \\ -1 & 2 & -3 \end{bmatrix}, \quad A_B^{-1}b = \begin{bmatrix} 30 \\ 10 \\ 10 \end{bmatrix},$$
$$c_B A_B^{-1} b = 20, \text{ and } \quad \bar{c} = c_B A_B^{-1} A_N - c_N = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}.$$

(e) The first iteration is

$$
\begin{array}{cccccc|c}
-2 & 1 & -1 & 0 & 0 & 0 & 0 \\
\hline
3 & 1 & 1 & 1 & 0 & 0 & 60 \\
\boxed{1} & -1 & 2 & 0 & 1 & 0 & 10 \\
1 & 1 & -1 & 0 & 0 & 1 & 20
\end{array}
\quad \rightarrow \quad
\begin{array}{cccccc|c}
0 & -1 & 3 & 0 & 2 & 0 & 20 \\
\hline
0 & 4 & -5 & 1 & -3 & 0 & 30 \\
1 & -1 & 2 & 0 & 1 & 0 & 10 \\
0 & 2 & -3 & 0 & -1 & 1 & 10
\end{array}
$$

$A_B^{-1}A_N$ is still in the constraint rows of nonbasic columns (5, 2, and 3), $A_B^{-1}b$ is still the RHS column, $c_B A_B^{-1}b$ is still in the right-top corner, and $\bar{c}$ is still in the objective row of nonbasic columns (5, 2, and 3).

(f) We run one more iteration to get

$$
\begin{array}{cccccc|c}
0 & -1 & 3 & 0 & 2 & 0 & 20 \\
\hline
0 & 4 & -5 & 1 & -3 & 0 & 30 \\
1 & -1 & 2 & 0 & 1 & 0 & 10 \\
0 & \boxed{2} & -3 & 0 & -1 & 1 & 10
\end{array}
\quad \rightarrow \quad
\begin{array}{cccccc|c}
0 & 0 & \frac{3}{2} & 0 & \frac{3}{2} & \frac{1}{2} & 25 \\
\hline
0 & 0 & 1 & 1 & -1 & -2 & 10 \\
1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 15 \\
0 & 1 & -\frac{3}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & 5
\end{array}
$$

The unique optimal solution to the original problem is $x^* = (x_1^*, x_2^*, x_3^*) = (15, 5, 0)$. The corresponding objective value is $z^* = 25$.

2. (a) We first convert the original problem to

$$
\begin{aligned}
z^* = \min \quad & 4x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 8 \\
& x_2 \leq 5 \\
& x_1 - x_2 \leq 4 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

so that all variables are nonnegative. Then we run one iteration to get

$$
\begin{array}{ccccc|c}
-4 & 1 & 0 & 0 & 0 & 0 \\
\hline
2 & 1 & 1 & 0 & 0 & 8 \\
0 & \boxed{1} & 0 & 1 & 0 & 5 \\
1 & -1 & 0 & 0 & 1 & 4
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
-4 & 0 & 0 & -1 & 0 & -5 \\
\hline
2 & 0 & 1 & -1 & 0 & 3 \\
0 & 1 & 0 & 1 & 0 & 5 \\
1 & 0 & 0 & 1 & 1 & 9
\end{array}
$$

An optimal solution to the new problem is $(x_1', x_2') = (0, 5)$. Therefore, an optimal solution to the original problem is $(x_1^*, x_2^*) = (0, -5)$ with objective value is $z^* = -5$.

(b) For the first basis $B = \{x_3, x_4, x_5\}$, the basic feasible solution is $x^0 = (0, 0, 8, 5, 4)$, which corresponds to the point $(0, 0)$ on Figure 1. For the second basis $B = \{x_3, x_2, x_5\}$, the basic feasible solution is $x^0 = (0, 5, 3, 0, 9)$, which corresponds to the point $(0, -5)$ on Figure 1.

3. (a) Let's change the objective function to $\max\ 2x_1 + 3x_2$. We then run two iterations to get

$$
\begin{array}{cccc|c}
-2 & -3 & 0 & 0 & 0 \\
\hline
1 & 2 & 1 & 0 & 6 \\
\boxed{2} & 1 & 0 & 1 & 8
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc|c}
-\frac{1}{2} & 0 & \frac{3}{2} & 0 & 9 \\
\hline
\frac{1}{2} & 1 & \frac{1}{2} & 0 & 3 \\
\boxed{\frac{3}{2}} & 0 & -\frac{1}{2} & 1 & 5
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc|c}
0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{32}{3} \\
\hline
0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \\
1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{10}{3}
\end{array}
$$

An optimal solution to the original problem is $(x_1^*, x_2^*) = (\frac{4}{3}, \frac{10}{3})$ with objective value is $z^* = \frac{32}{3}$. The optimal solution is identical to the one we found in class.

(b) The route we go through is depicted in Figure 2. In the first iteration we move from $x^0 = (0, 0)$ to $x^1 = (0, 3)$ and then in the first iteration from $x^1 = (0, 3)$ to $x^2 = (\frac{4}{3}, \frac{10}{3})$.

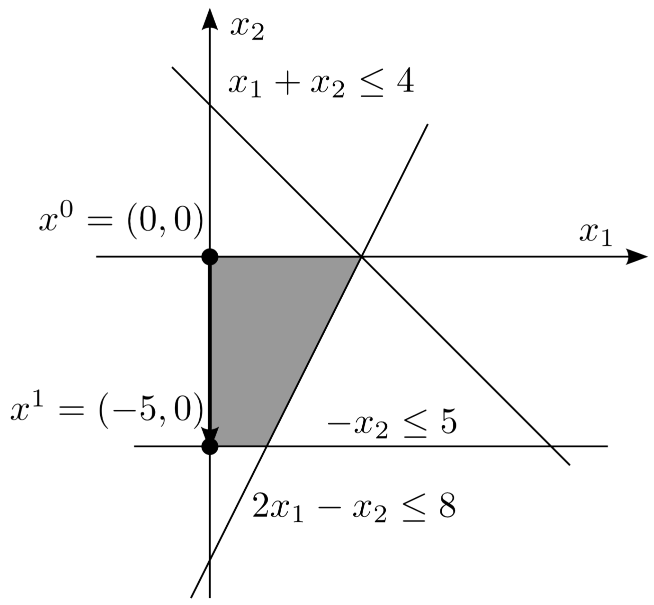

Figure 1: The solution process for Problem 2

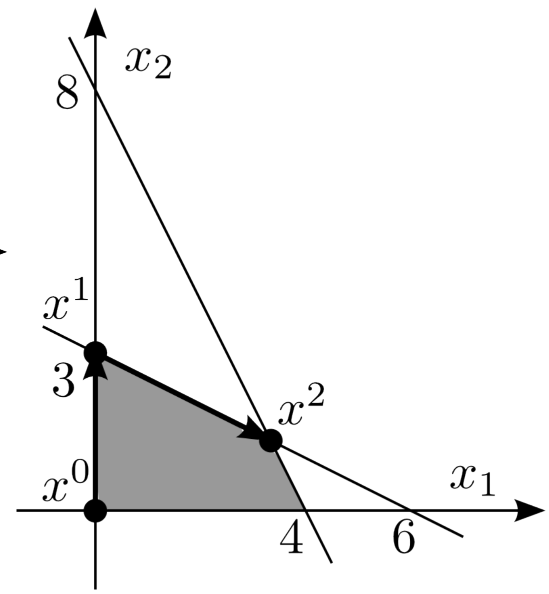

Figure 2: The solution process for Problem 3

4. To formulate this problem, we need an enough number of decision variables to completely describe an officer schedule. More precisely, the set of decision variables we define should have the following property: If we have the values for all the variables, we have the number of officers working in each day. If all the officers work five consecutive days, we need only seven variables to make a complete description. In this problem we need more because officers are not required to work consecutively. To make our notation concise, we label Sunday as day 1, Monday as day 2, ..., and Saturday as day 7. Then let

$$x_{ij} = \text{number of officers off on days } i \text{ and } j,\ i = 1, ..., 7, j = i + 1, ..., 7.$$

This definition actually defines the following 21 variables: x_{12}, x_{13}, ..., x_{17}, x_{23}, x_{24}, ..., x_{27}, x_{34}, ..., x_{37}, ..., x_{67}. For example, x_{12} is the number of officers that are off on Sunday and Monday, x_{13} is the number of officers that are off on Sunday and Tuesday, and so on. This set of variables gives us a complete description of the officer schedule.

The objective is minimizing the number of officers whose days off are not consecutive, or equivalently, maximizing the number of officers whose days off are consecutive. Therefore, we maximize $x_{12} + x_{23} + \cdots + x_{67} + x_{17}$. For Sunday, we need at least 18 officers, which means we may have at most 12 officers off on Sunday. This is achieved by having $x_{12} + x_{13} + \cdots + x_{17} \leq 12$. Similar arguments give us the constraints for the other six days. Finally, the total number of officers is 30, so we should have $x_{12} + x_{13} + \cdots + x_{67} = 30$. The complete formulation is

$$\begin{array}{rlll} z^* = & \max & x_{12} + x_{23} + x_{34} + x_{45} + x_{56} + x_{67} + x_{17} & \\ & \text{s.t.} & x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} \leq 12 & \text{(Number of officers off on Sunday)} \\ & & x_{12} + x_{23} + x_{24} + x_{25} + x_{26} + x_{27} \leq 12 & \text{(Number of officers off on Monday)} \\ & & x_{13} + x_{23} + x_{34} + x_{35} + x_{36} + x_{37} \leq 6 & \text{(Number of officers off on Tuesday)} \\ & & x_{14} + x_{24} + x_{34} + x_{45} + x_{46} + x_{47} \leq 5 & \text{(Number of officers off on Wednesday)} \\ & & x_{15} + x_{25} + x_{35} + x_{45} + x_{56} + x_{57} \leq 14 & \text{(Number of officers off on Thursday)} \\ & & x_{16} + x_{26} + x_{36} + x_{46} + x_{56} + x_{67} \leq 9 & \text{(Number of officers off on Friday)} \\ & & x_{17} + x_{27} + x_{37} + x_{47} + x_{57} + x_{67} \leq 2 & \text{(Number of officers off on Saturday)} \\ & & \sum_{i=1}^{7} \sum_{j=i+1}^{7} x_{ij} = 30 & \text{(Total number of officers)} \\ & & x_{ij} \geq 0 \quad \forall\ i = 1, ..., 7, j = i + 1, ..., 7. & \end{array}$$

5. (a) We need one slack variable for each functional constraint. Therefore, we need m constraints.

   (b) Let $y = (y_1, y_2, ..., y_m)$ be the slack variables, the standard form can be expressed as

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax + Iy = b \\ & x, y \geq 0. \end{aligned}$$

   If we select y to be the basis, we know x is going to be 0 and thus for the associated basic solution we solve $Iy = b$, which results in $y = b \geq 0$. Therefore, this basic solution is feasible.