Operations Research, Spring 2013
Suggested Solution for Homework 05

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. For me, Ling-Chieh Kung, I have no midterm exam between April 15th and 19th.

2. We run one iteration to get

$$\begin{array}{rrrrr|r} -1 & -1 & 0 & 0 & 0 & 0 \\ \hline \boxed{1} & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 1 \end{array} \quad \to \quad \begin{array}{rrrrr|r} 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 1 & \boxed{1} & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 & 1 & 0 \end{array}$$

The optimal solution to the original problem is $(x_1^*, x_2^*, x_3^*) = (1, 0, 0)$. The corresponding objective value is 1. Since there is one nonbasic variable x_2 having 0 in the objective row, we may enter it and get the following tableau:

$$\begin{array}{rrrrr|r} 0 & 0 & 1 & 1 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 1 \end{array}$$

Therefore, another optimal solution to the original problem is $x^* = (x_1^*, x_2^*, x_3^*) = (0, 1, 0)$. The corresponding objective value is 1.

3. (a) Because all the reduced costs are nonnegative, this is an optimal tableau. As there is an optimal solution, the problem is not unbounded.

   Note. Even though the third column contains only nonpositive numbers in the constraint rows, that just means this is an unbounded direction. It does not imply that the problem is unbounded.

   (b) Yes, there are multiple optimal solutions. First, we know this tableau corresponds to an optimal solution. Second, we find that there is a nonbasic variable whose reduced cost is zero. If we try to enter this variable, we then realize that we can indeed move for a positive distance. In fact, the direction is unbounded. Combining all the above conditions, we conclude that there are multiple optimal solutions.

   (c) There is only one optimal basic feasible solution (according to this tableau). The current optimal solution associated with this tableau is basic. However, all other optimal solution that can be found along the direction are not basic.

4. We run one iteration as below. Since the first column has a negative number in the objective row and nonpositive numbers in the first and second row, we know the problem is unbounded.

$$\begin{array}{rrrr|r} 0 & -2 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 0 & 4 \\ -1 & \boxed{1} & 0 & 1 & 1 \end{array} \quad \to \quad \begin{array}{rrrr|r} -2 & 0 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 1 & 5 \\ -1 & 1 & 0 & 1 & 1 \end{array}$$

5. First we generate the Phase-I program

$$\begin{array}{lrcrcrcrcrcl} \min & & & & & & + & x_4 & + & x_5 & & \\ \text{s.t.} & x_1 & + & 2x_2 & - & x_3 & + & x_4 & & & = & 6 \\ & 2x_1 & + & 3x_2 & & & & & + & x_5 & = & 4 \\ & & & & & & & & & x_i & \geq & 0 \quad \forall\, i = 1, ..., 5. \end{array}$$

We then prepare the initial tableau by fixing the objective row

$$
\begin{array}{ccccc|c}
0 & 0 & 0 & -1 & -1 & 0 \\
\hline
1 & 2 & -1 & 1 & 0 & x_4 = 6 \\
2 & 3 & 0 & 0 & 1 & x_5 = 4
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
3 & 5 & -1 & 0 & 0 & 10 \\
\hline
1 & 2 & -1 & 1 & 0 & x_4 = 6 \\
2 & 3 & 0 & 0 & 1 & x_5 = 4
\end{array}
$$

and run two simplex iterations and get

$$
\begin{array}{ccccc|c}
3 & 5 & -1 & 0 & 0 & 10 \\
\hline
1 & 2 & -1 & 1 & 0 & x_4 = 6 \\
\boxed{2} & 3 & 0 & 0 & 1 & x_5 = 4
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{cccc|c}
0 & \frac{1}{2} & -1 & 0 & 4 \\
\hline
0 & \frac{1}{2} & -1 & 1 & x_4 = 4 \\
1 & \boxed{\frac{3}{2}} & 0 & 0 & x_1 = 2
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc|c}
-\frac{1}{3} & 0 & -1 & 0 & \frac{10}{3} \\
\hline
\frac{-1}{3} & 0 & -1 & 1 & x_4 = \frac{10}{3} \\
\frac{2}{3} & 1 & 0 & 0 & x_2 = \frac{4}{3}
\end{array}
$$

This is the optimal tableau (for the Phase-I program), but we still have an artificial variable (x_4) in the basis. Therefore, we conclude that the original problem is infeasible. You may draw a graph to verify this.

6. We define

$$x_i = \begin{cases} 1 & \text{if player } i \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}, \ i = 1, ..., 7$$

as our decision variables. We also define $D = (3, 2, 2, 1, 3, 3, 1)$ as the vector of defense levels and $R = (1, 3, 2, 3, 3, 2, 2)$ as the vector of rebounding levels.

The complete formulation is

$$
\begin{aligned}
\max \quad & \textstyle\sum_{i=1}^{7} D_i x_i \\
\text{s.t.} \quad & \textstyle\sum_{i=1}^{7} x_i = 5 \\
& x_1 + x_3 + x_5 + x_7 \geq 2 \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq 2 \\
& x_2 + x_4 + x_6 \geq 1 \\
& \textstyle\sum_{i=1}^{7} R_i x_i \geq 10 \\
& x_3 + x_6 \leq 1 \\
& 2x_1 \leq x_4 + x_5 \\
& x_2 + x_3 \geq 1 \\
& x_i \in \{0, 1\} \quad \forall\, i = 1, ..., 7.
\end{aligned}
$$

The objective function maximizes the total defense level. The first constraint chooses five starting players. The second to the four constraints ensure that we have enough players that can play guard, forward, and center. The fifth constraint ensures that the average rebounding level is at least 2. The sixth constraint ensures that "if player 3 starts, then player 6 cannot start." The seventh constraint ensures that "if player 1 starts, then players 4 and 5 must both start." The eighth constraint ensures that "Either player 2 or player 3 must start."

7. We define

$$x_i = \text{tons of water processed in station } i,\ i = 1, ..., 3, \text{ and}$$

$$y_i = \begin{cases} 1 & \text{if station } i \text{ is used} \\ 0 & \text{otherwise} \end{cases}, \ i = 1, ..., 3$$

as our decision variables. We also define $F = (100000, 60000, 40000)$ as the fixed cost vector for stations, $C = (20, 30, 40)$ as the variable cost vector for stations, $P = (80000, 50000)$ as the vector of pollutants to be removed for pollutants, and $R = \begin{bmatrix} 0.4 & 0.3 \\ 0.25 & 0.2 \\ 0.2 & 0.25 \end{bmatrix}$ as the matrix of pollutants removal for stations and pollutants. Let M_i be very large numbers for a while, $i = 1, ...,$, the complete formulation is

$$
\begin{array}{lll}
\min & \sum_{i=1}^{3} F_i y_i + \sum_{i=1}^{3} C_i x_i & \\
\text{s.t.} & \sum_{i=1}^{3} R_{ij} x_i \geq P_j & \forall\, j = 1, 2 \\
& x_i \leq M_i y_i & \forall\, i = 1, ..., 3 \\
& x_i \geq 0 & \forall\, i = 1, ..., 3 \\
& y_i \in \{0, 1\} & \forall\, i = 1, ..., 3.
\end{array}
$$

The first constraint is for us to remove the required amount of both pollutants. The second constraint sets up the binary variables: When $x_i > 0$, we have $y_i = 1$. The objective is to minimize the total fixed and variable costs.

Now we find appropriate values for M_i, i.e., an upper bound of x_i. These upper bounds can be found by treating station i as the only station we build. For example, if we only build station 1, then we need $\frac{P_1}{R_{11}}$ tons of water processed in station 1 to meet the pollutant removal constraint for pollutant 1. Similarly, we need $\frac{P_2}{R_{12}}$ tons of water for pollutant 2. We thus need to process $\max\left\{\frac{P_1}{R_{11}}, \frac{P_2}{R_{12}}\right\}$ tons of water in station 1 (if we only build it). This quantity then become an upper bound of x_1. In general, we can define $M_i = \max_{j=1,2}\left\{\frac{P_j}{R_{ij}}\right\}$ as an upper bound of x_i. Then we can replace the constraint $x_i \leq My_i$ by $x_i \leq M_i y_i$. With the problem parameters, we have $M_1 = 200000$, $M_2 = 320000$, and $M_3 = 400000$.

8. Let z be a binary variable such that

$$
z = \begin{cases} 0 & \text{if } x + y \leq 3 \text{ is satisfied and} \\ 1 & \text{if } 2x + 5y \leq 12 \text{ is satisfied} \end{cases}.
$$

Let M_1 and M_2 be upper bounds of $x + y - 3$ and $2x + 5y - 12$, respectively, then the following two constraints

$$
\begin{aligned}
x + y - 3 &\leq M_1 z \\
2x + 5y - 12 &\leq M_2(1 - z)
\end{aligned}
$$

ensures that at least one of $x + y \leq 3$ and $2x + 5y \leq 12$ is satisfied. The condition that both x and y are integers is not important. Note that because there is no information regarding the possible values of x and y, there is no way for us to find a specific values for M_1 and M_2.

9. First, note that "if $x \leq 2$ then $y \leq 3$" is equivalent to "$x > 2$ or $y \leq 3$". Before we apply the technique of modeling "either-or" requirements, note that it is not allowed to have strict inequalities in an LP or IP formulation. Therefore, we must apply the condition that x is an integer to convert $x > 2$ into a weak inequality. To do this, note that $x > 2$ is equivalent to $x \geq 3$ if x is an integer. Therefore, all we need to do is to write constraints so that "$x \geq 3$ or $y \leq 3$". Let z be a binary variable such that

$$
z = \begin{cases} 0 & \text{if } x \geq 3 \text{ is satisfied and} \\ 1 & \text{if } y \leq 3 \text{ is satisfied} \end{cases}.
$$

Let M_1 and M_2 be upper bounds of $3 - x$ and $y - 3$, respectively, then the following two constraints

$$
\begin{aligned}
3 - x &\leq M_1 z \\
y - 3 &\leq M_2(1 - z)
\end{aligned}
$$

ensures that at least one of $x \geq 3$ and $y \leq 3$ is satisfied, i.e., if $x \leq 2$ then $y \leq 3$. Note that because there is no information regarding the possible values of x and y, there is no way for us to find a specific values for M_1 and M_2.