# IM2010: Operations Research
# Introduction to Model Building
# (Chapter 1)

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# Introduction

- **Mathematical modeling** is a way of **abstracting** a physical problem into a **model** with symbols and formulas.
- Let's see the following example.
  - I have three used books to sell in a second-hand market.
  - I need to bring them to the market.
  - But I may carry at most 5 kg.

| Book | Price (NT$) | Weight (kg) |
|---|---:|---:|
| Calculus | 400 | 4 |
| Statistics | 200 | 1.5 |
| Operations Research | 300 | 3 |

  - Which book(s) should I bring?

# Introduction

- To decide what to do, we will solve a **mathematical model**.
- A mathematical model is often called a **mathematical program** in the world of Operations Research.
- We will study all kinds of mathematical programs:
  - Their properties.
  - Ways of finding the solution (i.e., algorithms).
  - Managerial insights.
  - Beauty of mathematics.
- We will be familiar with the **process** of solving a problem with mathematical programming.

# Road map

- **Introduction to (mathematical) modeling**.
- Foundations of mathematical programming.
- Four steps of solving a problem.

# Decision making

- When we need to **make a decision**:
  - What are we going to decide?
  - What do we want?
  - What kind of limitations we are facing?

# Decision making

- Suppose I need to decide what to do tonight. I may watch a movie, sleep, work on slides, or writing papers.
  - Decision: What to do.
  - Objective: Make me feel happy.
  - Limitations: Survive.

http://yourmoviesdownload.net/

http://www.123rf.com/

# Optimization problems

- We use a mathematical program (or mathematical model) to **formulate** our problem.
- In general, all the problems we face in our life can be formulated as **optimization problems**:
  - Deciding what to do tonight to make myself as happy as possible.
  - Choosing a restaurant to best meet my taste and budget.
  - Determining the route of sending mails to spend the least time.
  - Selecting a price to maximize my sales revenue.
  - Making a production plan to minimize my total cost.
- What are the components in an optimization problem?

# Optimization problems

- In an optimization problem, there are:
  - **Decision variables**.
  - **The objective function**.
  - **Constraints**.
- Let's understand them by formulating the problem of which books to carry.

- Question: Choose some books to carry to earn as much money as possible. But I can carry at most 5 kg.

| Book | Price (NT$) | Weight (kg) |
|---|---|---|
| Calculus | 400 | 4 |
| Statistics | 200 | 1.5 |
| Operations Research | 300 | 3 |

- Step 1: Use **symbols** and **functions** to describe the problem.
  - Let $B = \{c, s, o\}$ be the set of books. Let $p(x)$ be the price of book $x$ and $w(x)$ be the weight of book $x$, $x \in B$.
  - E.g., $p(c) = 400$ and $w(s) = 1.5$.

# Formulating a problem

- Step 2: Describe **all possible** actions by ignoring constraints.
  - We may choose $c$ only. We may choose $c$ and $s$. We may choose $c$ and $o$. We may choose...
  - In general, we are choosing a subset of $B$.
- Step 3: Define the **decision variables**.
  - Let $X$ be the set of books that I am carrying to the market.
  - Be specific, clear, precise, and complete!
  - Bad definition 1: Let $X$ be books.
  - Bad definition 2: $X =$ books.
  - Not very good definition: Let $X$ be the books I select.

# Formulating a problem

- Step 4: Write down a **maximization** or **minimization objective function**.
  - I want to earn as much money as possible.
  - So I want to maximize my sales revenue.
  - $\max \sum_{x \in X} p(x)$.
- Step 5: Write down **constraints** as **equalities** or **inequalities**.
  - I can carry at most 5 kg: $\sum_{x \in X} w(x) \leq 5$.
  - I cannot sell what I do not have: $X \subseteq S$.

# Formulating a problem

- The complete formulation:

$$\begin{aligned} \max \quad & \sum_{x \in X} p(x) \\ \text{s.t.} \quad & \sum_{x \in X} w(x) \leq 5 \\ & X \subseteq S. \end{aligned}$$

  - The decision variable is a set.
  - There is one maximization objective function.
  - There are two constraints.

# An alternative formulation

- In designing and running algorithms, typically it is easier to not using a set as a decision variable.
- Let's label Calculus as book 1, Statistics as book 2, and OR as book 3. Then we may define $p_i$ and $w_i$ as the price and weight of book $i$, $i = 1, ..., 3$.
- Furthermore, let

$$x_i = \begin{cases} 1 & \text{if I carry book } i \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 3$$

be our decision variables.

# An alternative formulation

- The first attampt of an alternative formulation:

$$\begin{aligned} \max \quad & \sum_{i=1}^{3} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{3} w_i x_i \leq 5. \end{aligned}$$

Is that all?

# An alternative formulation

- There is one more constraint: We cannot split the book:

$$x_i \in \{0,1\} \quad \forall i = 1, ..., 3.$$

- The complete formulation:

$$\begin{aligned} \max \quad & \sum_{i=1}^{3} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{3} w_i x_i \leq 5 \\ & x_i \in \{0,1\} \quad \forall i = 1, ..., 3. \end{aligned}$$

# Some remarks

- The problem is an example of the **knapsack** problem, one of the most fundamental problem in Computer Science.
- In general, a decision variable can be a scalar, a vector, a matrix, a set, etc.
  - In this course, almost all variables are scalars.
- In most problems, there is only one objective function.
  - Either maximization or minimization.
- In the subject of multi-objective optimization, there can be multiple objective functions.
- There can be any number of constraints.

# Some remarks

- For a single problem, there are always **multiple** ways of formulating it.
  - In this course, you will need to formulate many problems.
  - While there are some differences among different formulations (some are **easier to solve**), you do not need to worry about that.
  - Just make your formulation **correct**, precise, and understandable.

# Summary

- Mathematical programming (modeling) is the entire process of
  - **formulating** a physical problem with mathematical terms,
  - **solving** the mathematical program,
  - **interpreting** the results, and
  - facilitating **decision making**.
- We need to
  - get experience in formulating problems,
  - study how to solve a program, and
  - practice to draw (managerial) insights from numbers.

# Road map

- Introduction to modeling.
- **Foundations of mathematical programming**.
- Four steps of solving a problem.

# Mathematical programming

- In general, a mathematical program (MP) can be expressed as

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, ..., m \\ & x \in X \subseteq \mathbb{R}^n. \end{aligned}$$

  - $x \in \mathbb{R}^n$ is the decision vector (set of decision variables).
  - $f(x)$ is the objective function.
  - $g_i(x) \leq 0$, the $i^{\text{th}}$ constraint, imposes a functional restriction on $x$.
  - $X$ is a nonfunctional restriction on $x$.

# Mathematical programming

- The formats of $f(\cdot)$, $g_i(\cdot)$s, and $X$ in a mathematical program

$$\begin{array}{rl} \min & f(x) \\ \text{s.t.} & g_i(x) \leq 0 \quad \forall i = 1, ..., m \\ & x \in X \subseteq \mathbb{R}^n. \end{array}$$

  categorize mathematical programs into several classes.
  - Linear programs.
  - Convex programs.
  - Nonlinear programs.
  - Linear integer programs.
  - Convex integer programs.
  - Nonlinear integer programs.

# Linear programming

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, ..., m \\ & x \in X \subseteq \mathbb{R}^n. \end{aligned}$$

- When $f(\cdot)$ and $g_i(\cdot)$s are all **linear** functions, i.e.,

  $$f(x) = c_0 + c_1 x_1 + \cdots + c_n x_n,$$

  and

  $$g_i(x) = a_{i0} + a_{i1} x_1 + \cdots + a_{in} x_n \quad \forall i = 1, ..., m,$$

  and $X = \mathbb{R}^n$, the mathematical program is a **linear program (LP)**.

# Linear programming

- An example:

$$\begin{aligned} \min \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 8 \\ & 4x_1 - x_2 \geq 6. \end{aligned}$$

- Regarding linear programming:
  - The most fundamental mathematical programming.
  - Efficient algorithms exist and are widely implemented.
  - Can always be solved efficiently.
  - One of the most applicable in business.

# Convex programming

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, ..., m \\
& x \in X \subseteq \mathbb{R}^n
\end{aligned}
$$

- When $f(\cdot)$ and $g_i(\cdot)$s are all **convex** functions and $X = \mathbb{R}^n$, the mathematical program is a **convex program (CP)**.
  - What is a convex function?

# Convex sets

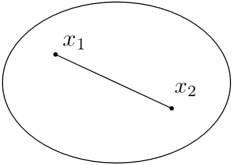
- We need to define **convex sets** and **convex functions**:

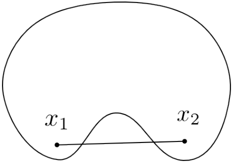
**Definition 1 (Convex sets)**

*A set $F$ is convex if*

$$\lambda x_1 + (1-\lambda)x_2 \in F$$

*for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.*



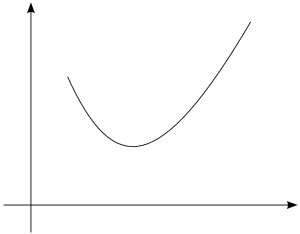
# Convex functions

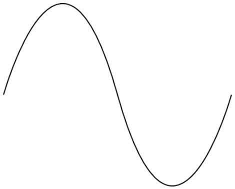
**Definition 2 (Convex functions)**

*For a convex set $F$, a function $f(\cdot)$ is convex on $F$ if*

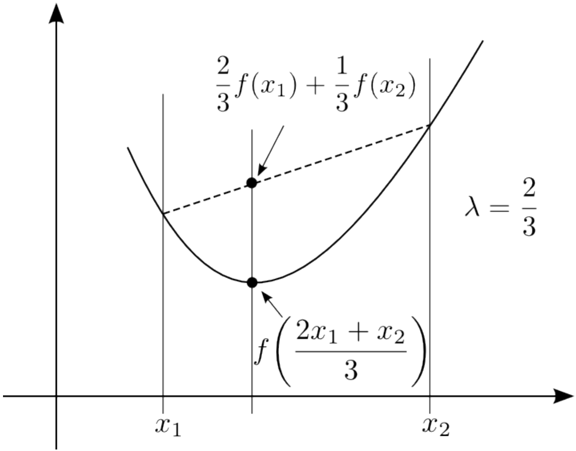
$$f\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

*for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.*





# Convex functions



# Some examples

- Convex sets?
  - $X_1 = [10, 20]$.
  - $X_2 = (10, 20)$.
  - $X_3 = \mathbb{N}$.
  - $X_4 = \mathbb{R}$.
  - $X_5 = \{(x, y) | x^2 + y^2 \leq 4\}$.
  - $X_6 = \{(x, y) | x^2 + y^2 \geq 4\}$.
- Convex functions?
  - $f_1(x) = x + 2, x \in \mathbb{R}$.
  - $f_2(x) = x^2 + 2, x \in \mathbb{R}$.
  - $f_3(x) = \sin(x), x \in [0, 2\pi]$.
  - $f_4(x) = \sin(x), x \in [\pi, 2\pi]$.
  - $f_5(x) = \log(x), x \in (0, \infty)$.
  - $f_6(x, y) = x^2 + y^2, (x, y) \in \mathbb{R}^2$.

# Convex programming

- An example:

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 9 \\ & e^{-x_1} \leq 2. \end{aligned}$$

- Regarding convex programming:
  - A super class of linear programming.
  - Efficient algorithms exist and are widely implemented.
  - Can usually be solved efficiently.
  - One of the most applicable in science, engineering, and Economics. Also used a lot in business.

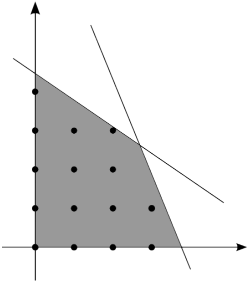
# Nonlinear programming

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i = 1, ..., m \\ & x \in X \subseteq \mathbb{R}^n. \end{aligned}$$

- When $X = \mathbb{R}^n$ and **at least** one of $f(\cdot)$ and $g_i(\cdot)$s is **not linear**, the mathematical program is a **nonlinear program (NLP)**.
  - LP $\subset$ CP, CP $\nsubseteq$ NLP, and LP $\cap$ NLP $= \emptyset$.
  - Typically cannot be solved efficiently.

# Linear integer programming

- When $f(\cdot)$ and $g_i(\cdot)$s are all linear functions and **at least** one variable $x_i \in X_i \subseteq \mathbb{Z}$ (and thus $X \neq \mathbb{R}^n$), the mathematical program is a linear **integer** program (LIP).



  - Typically referred to as **integer programming (IP)**.
  - Widely used in practice (mainly in business).
  - In general not easy to solve.
  - Satisfactory methods exist for problems in reasonable scales.

# Convex/nonlinear integer programming

- When $f(\cdot)$ and $g_i(\cdot)$s are all convex functions and at least one variable $x_i \in X_i \subseteq \mathbb{Z}$ (and thus $X \neq \mathbb{R}^n$), the mathematical program is a convex integer program (CIP).
- When any of $f(\cdot)$ and $g_i(\cdot)$s is nonlinear and at least one variable $x_i \in X_i \subseteq \mathbb{Z}$ (and thus $X \neq \mathbb{R}^n$), the mathematical program is a nonlinear integer program (NLIP).
- Hard and call for future research.

# Summary

| | Linear | Convex | Nonlinear |
|---|---|---|---|
| Continuous | LP | CP | NLP |
| Discrete | LIP | CIP | NLIP |

- In this semester:
  - Linear programming: about five weeks.
  - Convex and nonlinear programming: about two weeks.
  - Linear integer programming: about one week.
  - Convex and nonlinear integer programming: No time for them!

# Road map

- Introduction to modeling.
- Foundations of mathematical programming.
- **Four steps of solving a problem**.

# Four steps of solving a problem

- Step 1: **Understand** the problem and **collect** relevant data.
- Step 2: **Formulate** the problem.
  - Step 2.1: Define the decision variables.
  - Step 2.2: Write down the objective functions.
  - Step 2.3: Write down the constraints.
- Step 3: **Solve** the problem.
  - Find an optimal solution.
  - Get the values of the decision variables.
- Step 4: **Interpret** the optimal solution.
  - Determining what to do.

# Four steps of solving a problem

- Question: Choose some books to carry to earn as much money as possible. But I am not able to carry all the books.
- Step 1:
  - Read the problem: "choose some books", "earn money", "cannot carry all".
  - Collect data: Can carry at most 5 kg; having three books:

| Book | Price (NT$) | Weight (kg) |
|---|---|---|
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| Operations Research | 300 | 3 |

# Four steps of solving a problem

- Step 2: Formulate the problem.

$$\begin{aligned} \max \quad & \sum_{i=1}^{3} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{3} w_i x_i \leq 5 \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 3. \end{aligned}$$

- Step 3: Solve the problem:
  - The optimal solution is $(x_1^*, x_2^*, x_3^*) = (0, 1, 1)$.
  - How to solve the problem with 100 books?
- Step 4: Interpret the optimal solution:
  - We should bring the Statistics and Operations Research textbooks.