IM2010: Operations Research
The Basics of Linear Programming (Chapter 3)

Ling-Chieh Kung

Department of Information Management
National Taiwan University

February 21, 2013

Introduction

- In the following weeks, we will study Linear Programming (LP).
  - It is used a lot in practice.
  - It also provides important theoretical properties.
  - It is good starting point for all OR subjects.
- We will study:
  - What kind of practical problems can be solved by LP.
  - How to formulate a problem as an LP.
  - How to solve an LP.
  - Any many more.

Road map

- **Terminology.**
- Basic properties.
- The graphical approach.

Introduction

- A **linear program** (LP) is a mathematical program whose objective function and constraints are all **linear** and variables are all **fractional**.
  - If nonlinear: Convex or Nonlinear Programming.
  - If not fractional (i.e., discrete): Integer Programming.

Basic elements of an LP

- In general, any LP can be expressed as

$$
\begin{aligned}
\min \quad & f(x) = \sum_{j=1}^{n} c_j x_j \quad (\textbf{objective function}) \\
\text{s.t.} \quad & g_i(x) = \sum_{j=1}^{n} A_{ij} x_j \le b_i \quad \forall i = 1, ..., m \quad (\textbf{constraints}) \\
& x_j \in \mathbb{R} \quad \forall j = 1, ..., n. \quad (\textbf{decision variable})
\end{aligned}
$$

  - A_{ij}s: the **constraint coefficients**.
  - b_is: the right-hand-side values (**RHSs**).
  - c_js: the **objective coefficients**.
  - As a convention, we will ignore $x_j \in \mathbb{R}$ in the sequel.

Transformation

- How about a maximization objective function?
  - $\max f(x) \Leftrightarrow \min -f(x)$.
- How about equality or greater-than-or-equal-to constraint?
  - $g_i(x) \geq b_i \Leftrightarrow -g_i(x) \leq -b_i$.
  - $g_i(x) = b_i \Leftrightarrow g_i(x) \leq b_i$ and $g_i(x) \geq b_i$ (which is $-g_i(x) \leq -b_i$.
- For example,

$$\begin{array}{rl} \max & x_1 - x_2 \\ \text{s.t.} & -2x_1 + x_2 \geq -3 \\ & x_1 + 4x_2 = 5. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \min & -x_1 + x_2 \\ \text{s.t.} & 2x_1 - x_2 \leq 3 \\ & x_1 + 4x_2 \leq 5 \\ & -x_1 - 4x_2 \leq -5. \end{array}$$

Matrix representation of an LP

- An LP can also be expressed in the **matrix representation**:

$$\begin{array}{rl} \min & cx \\ \text{s.t.} & Ax \leq b. \end{array}$$

  - $A \in \mathbb{R}^{m \times n}$: the **constraint matrix**.
  - $b \in \mathbb{R}^m$: the **RHS vector** (a column vector).
  - $c \in \mathbb{R}^n$: the **objective vector** (a row vector).
- For example,

$$\begin{array}{rl} \max & x_1 - x_2 \\ \text{s.t.} & -2x_1 + x_2 \geq 3 \\ & x_1 + 4x_2 = 5. \end{array} \Rightarrow A = \begin{bmatrix} 2 & -1 \\ 1 & -4 \\ -1 & 4 \end{bmatrix}, b = \begin{bmatrix} -3 \\ 5 \\ -5 \end{bmatrix}, c = \begin{bmatrix} -1 & 1 \end{bmatrix}.$$

- The matrix representation will be used a lot in this course.

Sign constraints

- For some reasons that will be clear in a couple weeks, we distinguish between two kinds of constraints:
  - **Sign constraints**: $x_i \geq 0$ or $x_i \leq 0$.
  - **Functional constraints**: all others.
- For a variable x_i:
  - It is **nonnegative** if $x_i \geq 0$.
  - It is **nonpositive** if $x_i \leq 0$.
  - It is **unrestricted in sign** (urs.) or **free** if there is no sign constraint for it.

Example

- Here is an example of LP:

$$
\begin{array}{rrcrcr}
\min & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
& x_1 & + & 2x_2 & \leq & 12 \\
& x_1 & - & 2x_2 & \geq & -8 \\
& x_1 & & & \geq & 0 \\
& & & x_2 & \geq & 0.
\end{array}
$$

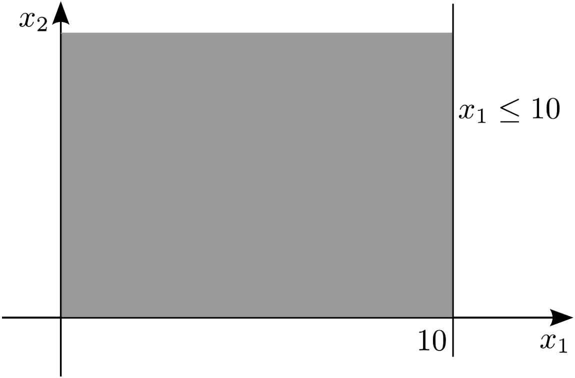
- The geometric representation may help.

Example

- With the sign constraints:

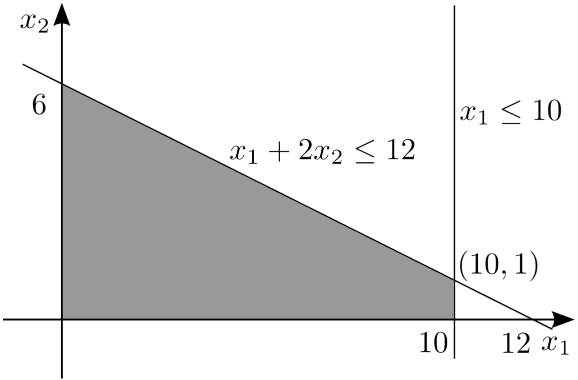
Example

- Adding $x_1 \leq 10$:



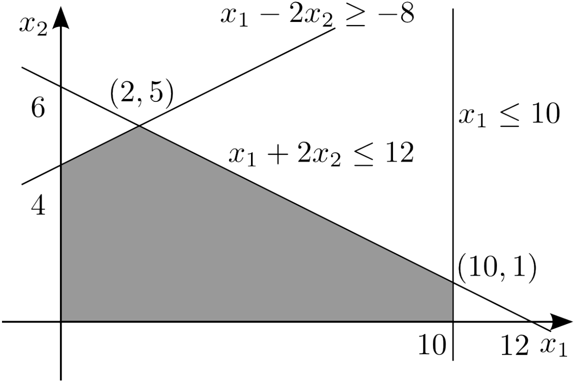
Example

- Adding $x_1 + 2x_2 \leq 12$:



Example

- Adding $x_1 - 2x_2 \geq -8$:



Example

- What is the matrix representation ($\min cx$ s.t. $Ax \leq b$) of

$$\begin{array}{rrcrcr} \min & 2x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & & & \leq & 10 \\ & x_1 & + & 2x_2 & \leq & 12 \\ & x_1 & - & 2x_2 & \geq & -8 \\ & x_1 & & & \geq & 0 \\ & & & x_2 & \geq & 0? \end{array}$$

  - $x = (x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.
  - $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 10 \\ 12 \\ 8 \\ 0 \\ 0 \end{bmatrix}$, and $c = \begin{bmatrix} 2 & 1 \end{bmatrix}$.

Feasible solutions

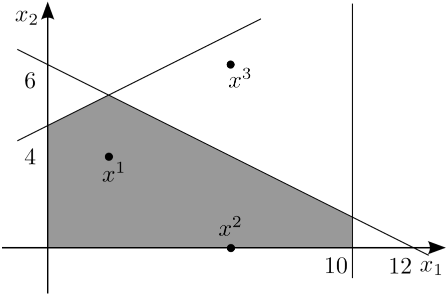
- For a linear program:
  - A **feasible solution** satisfies all the constraints.
  - An **infeasible solution** violates at least one constraints.

$$
\begin{array}{rrcrcr}
\min & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 12 \\
 & x_1 & - & 2x_2 & \geq & -8 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0
\end{array}
$$

- Feasible?
  - $x^1 = (2, 3)$.
  - $x^2 = (6, 0)$.
  - $x^3 = (6, 6)$.

Feasible solutions

- Graphically:



- Feasible?
  - $x^1 = (2, 3)$.
  - $x^2 = (6, 0)$.
  - $x^3 = (6, 6)$.

Feasible region and optimal solutions

- For a linear program:
  - The **feasible region** (or **feasible set**) is the set of feasible solutions.
  - An **optimal solution** is a feasible solution that optimizes (minimizes or maximizes) the optimal solution.
- The feasible region may be empty.
- The feasible region is unique.
- An optimal solution may not be unique.
  - There may be multiple optimal solutions or no optimal solution.

Strict constraints?

- An inequality is **strict** if the relationship is strictly greater than ($>$) or strictly less than ($<$).
  - E.g., $x_1 + x_2 > 5$.
- An inequality is **weak** if the relationship is weakly greater than ($\geq$) or weakly less than ($\leq$).
  - E.g., $x_1 + x_2 \geq 5$.
- An LP can only have **equalities** and **weak inequalities**.
  - With strict inequalities, is an optimal solution always attainable?
  - What is the optimal solution of

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & x > 0? \end{aligned}$$

Summary

- The decision variables, objective function, and constraints.
- Functional and sign constraints.
- Geometric representation.
- Feasible solutions and optimal solutions.

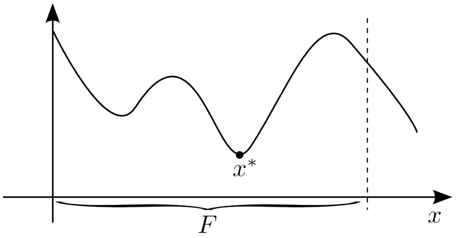
Road map

- Terminology.
- **Basic properties**.
- The graphical approach.

Global optima

- For a function $f(x)$ over a feasible region F, a point x^* is a **global minimum** if

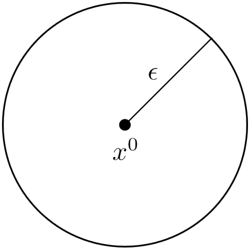
$$f(x^*) \leq f(x) \quad \forall x \in F.$$



  - There may be multiple global minima.
- x^* is a **global maximum** if $f(x^*) \geq f(x) \quad \forall x \in F$.
- A **global optimum** is either a minimum or a maximum.

Local optima

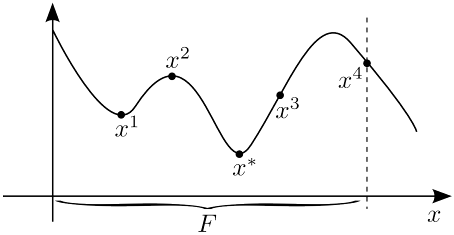
- Let $d(x, y) \equiv \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ be the Euclidean distance between two points x and $y \in \mathbb{R}^n$.
- Consider a point $x^0 \in \mathbb{R}^n$.
- We define $B(x^0, \epsilon) \equiv \{x | d(x, x^0) \leq \epsilon\}$ as a ball centered at x^0 with radius ϵ.



Local optima

- For a function $f(x)$ over a feasible region F, a point x' is a **local minimum** if for some $\epsilon > 0$ we have

$$f(x') \leq f(x) \quad \forall x \in B(x', \epsilon) \cap F.$$



- **Local maxima** and **local optima** are defined accordingly.

Local v.s. global optima

- Finding a local optimum is **easier** than finding a global one.
  - All we need to check is whether there is a feasible direction that improves the current solution.
  - This is called an **improving direction**.
  - If yes, move in that direction for some length.
  - Otherwise, we are at a local optimum.
- But in general, finding a local optimum is **not enough**.
- When is a local optimum also a global optimum?

Proposition 1

For a convex function over a convex feasible region, a local minimum is a global minimum.

Convex sets

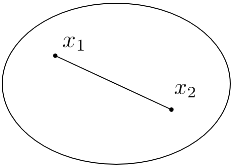
- Recall that we have defined convex sets and functions:

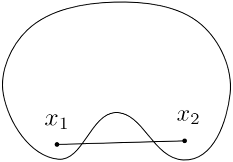
Definition 1 (Convex sets)

A set F is convex if

$$\lambda x_1 + (1-\lambda)x_2 \in F$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.





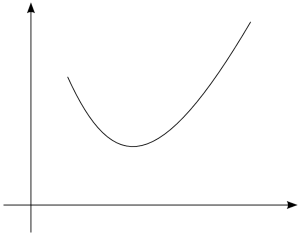
Convex functions

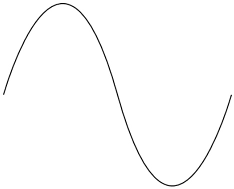
Definition 2 (Convex functions)

A function $f(\cdot)$ is convex if

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

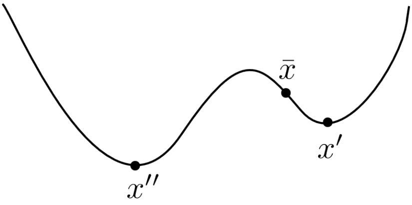
for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.





Local v.s. global optima

Proof. Let $f(\cdot)$ be a convex function over a convex feasible region F. Suppose a local min x' is not a global min and there exists x'' such that $f(x'') < f(x')$. Consider a small enough $\lambda > 0$ such that $\bar{x} = x' + \lambda(x'' - x')$ satisfies $f(\bar{x}) > f(x')$. Such $\bar{x}$ exists because x is a local min. Moreover, $\bar{x}$ is feasible because F is convex and x' and x'' are both feasible.



Local v.s. global optima

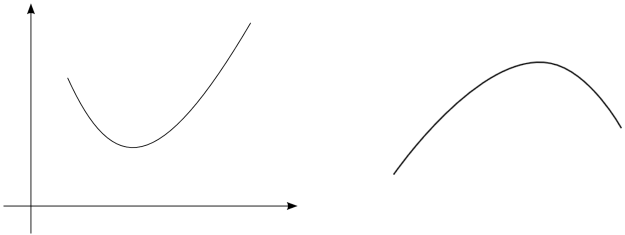
Proof (cont'd). Now, note that

$$
\begin{aligned}
f(\bar{x}) &= f\Big(\lambda x'' + (1-\lambda)x'\Big) \\
&> f(x') && (x' \text{ is a local min}) \\
&> \lambda f(x'') + (1-\lambda) f(x'), && (f(x'') < f(x'))
\end{aligned}
$$

which violates the fact that $f(\cdot)$ is convex. Therefore, by contradiction, the local min x must be a global min. $\square$

Local v.s. global optima

- Now we know if we minimize a convex function over a convex feasible region, a local minimum is a global minimum.
- If we **maximize a concave function** over a convex feasible region, a local maximum is a global maximum.
  - A function $f(\cdot)$ is **concave** if $-f(\cdot)$ is convex.



- What may happen if we **minimize a concave function**?

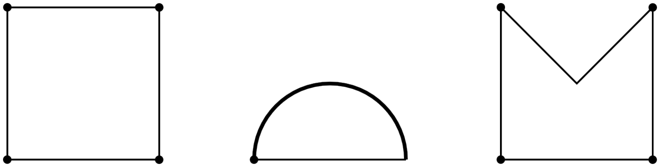
Extreme points

- We need to first define **extreme points** for a set:

Definition 3 (Extreme points)

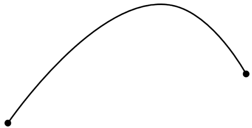
For a set S, a point x is an extreme point if there does not exist a three-tuple (x^1, x^2, λ) such that $x^1 \in S \setminus \{x\}$, $x^2 \in S \setminus \{x\}$, $\lambda \in (0, 1)$, and

$$x = \lambda x^1 + (1 - \lambda) x^2.$$



Extreme points and optimal solutions

- When we minimize a concave function over a convex feasible region, we only need to focus on the "**boundary**".
  - Mathematically, we focus on extreme points.



- Let's formalize the idea:

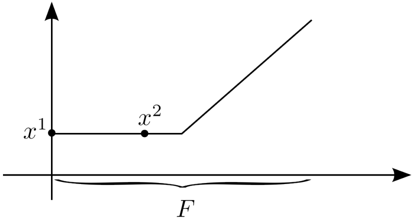
Proposition 2

For any concave function that has a global minimum, there exists a global minimum that is an extreme point.

Proof. Beyond the scope of this course. □

Warning!

- The proposition says "there exists a global minimum that is an extreme point."
- It is **not** "a global minimum must be an extreme point."
- For some problems, an extreme-point global minimum and a nonextreme-point global minimum may exist together!



- x^1 is an extreme-point global min.
- x^2 is a nonextreme-point global min.

Solving a linear program

- Now we know when we minimize $f(\cdot)$ over a convex feasible region F:
  - If $f(\cdot)$ is **convex**, search for a **local min**.
  - If $f(\cdot)$ is **concave**, search among the **extreme points** of F.
- How are these related to Linear Programming?
- We will show that, for any linear program:
  - The feasible region is convex.
  - The objective function is both convex and concave.
- Then the results will mean a lot to Linear Programming!

Solving a linear program

Proposition 3

The feasible region of a linear program is convex.

Proof. First, note that the feasible region of a linear program is the intersection of several half spaces (each one is determined by an inequality constraint) and hyperplanes (each one is determined by an equality constraint). It is trivial to show that half spaces and hyperplanes are always convex. It then remains to show that the intersection of convex sets are convex, which is left as a homework problem. □

Solving a linear program

Proposition 4

A linear function is both convex and concave.

Proof. To show that a function $f(\cdot)$ is convex and concave, we need to show that $f(\lambda x^1 + (1-\lambda)x^2) = \lambda f(x^1) + (1-\lambda)f(x^2)$, which is exactly the separability of linear functions: Let $f(x) = c \cdot x + b$ be a linear function, $c \in \mathbb{R}^n, b \in \mathbb{R}$, then

$$\begin{aligned} f\Big(\lambda x^1 + (1-\lambda)x^2\Big) &= c \cdot \Big(\lambda x^1 + (1-\lambda)x^2\Big) + b \\ &= \lambda(c \cdot x^1 + b) + (1-\lambda)(c \cdot x^2 + b) \\ &= \lambda f(x^1) + (1-\lambda)f(x^2). \end{aligned}$$

Therefore, a linear function is both convex and concave. $\square$

Solving a linear program

- In solving a linear program, we only need to search for a local minimum.
  - As long as we find a **feasible improving direction**, just move along that direction.
- Also, we only need to search among the extreme points of the feasible region.
  - We may **keep moving** until we reach the **end** of the feasible region.
- These two properties form the foundation of the **graphical approach** for solving two-dimensional linear programs.
- They also allow us to use the **simplex method** for solving n-dimensional linear programs.

Solving a linear program

- Linear programs are special.
  - Moving along any improving direction until we reach the end of the feasible region.
- Does this apply to nonlinear programs?
  - Is a local optimum always a global optimum?
  - Is there always an extreme point global optimum?
- Examples:

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 2 \\ & 2x_1 - x_2 \leq 2. \end{aligned}$$

$$\begin{aligned} \min \quad & x^3 - 3x^2 + 1 \\ \text{s.t.} \quad & x \geq -2. \end{aligned}$$

Road map

- Terminology.
- Basic properties.
- **The graphical approach**.
  - The steps.
  - The four types of LPs.

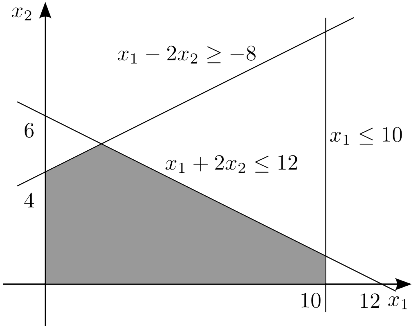
Graphical approach

- For linear programs with only **two** decision variables, we may solve them with the graphical approach.
- We will demonstrate this approach by solving the example

$$
\begin{array}{rrcrcr}
\max & 2x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & & & \leq & 10 \\
 & x_1 & + & 2x_2 & \leq & 12 \\
 & x_1 & - & 2x_2 & \geq & -8 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \geq & 0.
\end{array}
$$

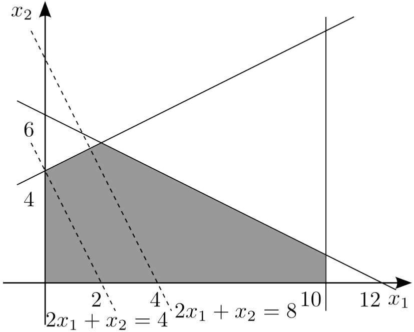
Graphical approach

- Step 1: Draw the feasible region.
  - Draw each constraint one by one, and then find the intersection.



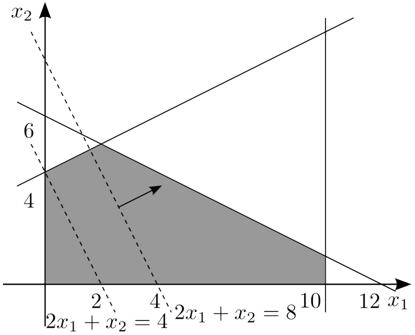
Graphical approach

- Step 2: Draw an **isocost line**.
  - A line such that all points on it result in **the same** objective value.
  - Also called the **isoprofit** line or **isoquant** line in some cases.



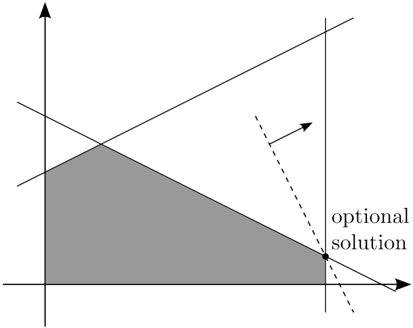
Graphical approach

- Step 3: Indicate the direction to push the isocost line.
  - The direction that **decreases**/increases the objective value for a **minimization**/maximization problem.
  - Typically depicted as a **vector perpendicular** to the isocost line.



Graphical approach

- Step 4: Push the isocost line to the "end" of the feasible region.
  - Stop when any further step makes all points on the isocost line infeasible.



Graphical approach

- Step 5: Identify the **binding constraints** at the optimal solution.

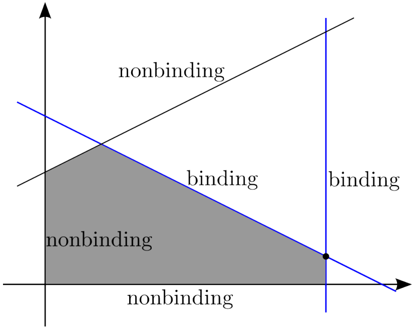
Definition 4

Let $g(\cdot) \leq b$ be an inequality constraint and x be a point. $g(\cdot)$ is binding at x if $g(x) = b$.

  - An inequality is **nonbinding** at a point if it is strict at that point.
  - An equality constraint is always binding at any feasible solution.
- Some examples:
  - $x_1 + x_2 \leq 10$ is binding at $(x_1, x_2) = (2, 8)$.
  - $2x_1 + x_2 \geq 6$ is nonbinding at $(x_1, x_2) = (2, 8)$.
  - $x_1 + 3x_2 = 9$ is binding at $(x_1, x_2) = (6, 1)$.

Graphical approach

- Step 5: Identify the binding constraints at the optimal solution.



Graphical approach

- Step 6: Set the binding constraints to equalities and then solve the linear system for an optimal solution.
  - In the example, the binding constraints are $x_1 \leq 10$ and $x_1 + 2x_2 \leq 12$. Therefore, we solve

$$\left[\begin{array}{cc|c} 1 & 0 & 10 \\ 1 & 2 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & 1 \end{array}\right]$$

    and obtain an optimal solution $(x_1^*, x_2^*) = (10, 1)$.
- Step 7: Plug in the optimal solution obtained into the objective function to get the associated objective value.
  - In the example, $2x_1^* + x_2^* = 21$.

Graphical approach: Summary

- Six steps:
  - Step 1: Feasible region.
  - Step 2: Isocost line.
  - Step 3: Direction to push (or the improving direction).
  - Step 4: Push!
  - Step 5: Binding constraints.
  - Step 6: Optimal solution.
- Make your graph clear and in the right scale to avoid mistakes.

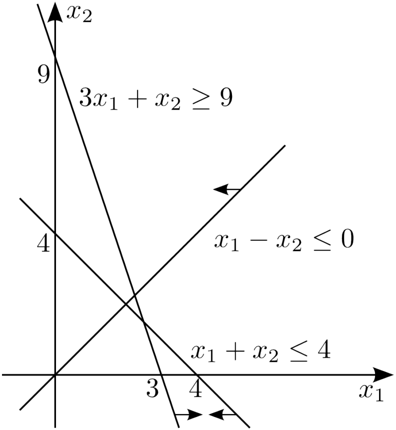
Four types of linear programs

- For any linear program, it must be one of the following:
  - Infeasible.
  - Unbounded.
  - Having a unique optimal solution.
  - Having multiple optimal solutions.

Infeasibility

- A linear program is **infeasible** if its feasible region is empty.

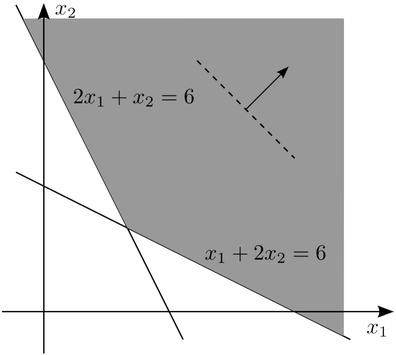
$$
\begin{array}{lrcrcl}
\min & 3x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
 & 3x_1 & + & x_2 & \geq & 9 \\
 & x_1 & - & x_2 & \leq & 0
\end{array}
$$



Unboundedness

- A linear program is **unbounded** if for any feasible solution, there is another feasible solution that is better.

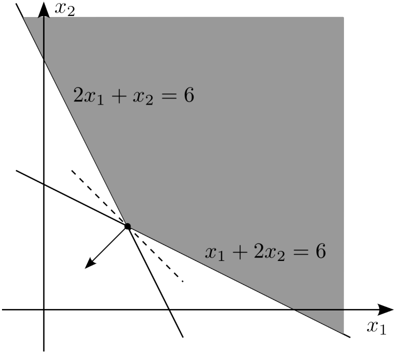
$$
\begin{array}{lrcrcr}
\max & x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\
 & 2x_1 & + & x_2 & \geq & 6
\end{array}
$$



Unboundedness

- Note that an unbounded feasible region **does not imply** an unbounded linear program!
  - Is it necessary?

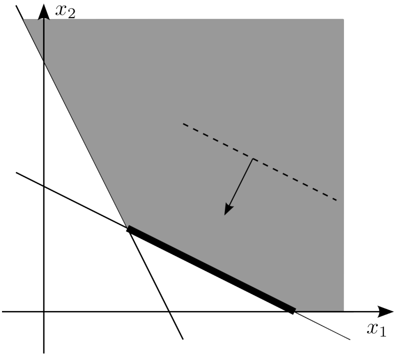
$$
\begin{array}{llcrcl}
\min & x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\
 & 2x_1 & + & x_2 & \geq & 6
\end{array}
$$



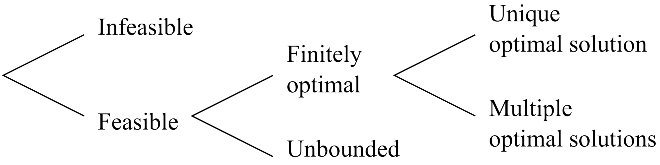
Multiple optimal solutions

- A linear program may have **multiple** optimal solutions.
  - If the slope of the isocost line is identical to that of one constraint, is it always the case that there are multiple optimal solutions?

$$
\begin{array}{llcrcl}
\min & x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \geq & 6 \\
 & 2x_1 & + & x_2 & \geq & 6 \\
 & & & x_2 & \geq & 0
\end{array}
$$



Four types of linear programs: Summary



- Nevertheless, in solving an LP (or any mathematical program), we only want to find **an** optimal solution, not all.
  - All we want is to make an optimal decision.