IM2010: Operations Research
The Simplex Method
(Chapter 4)

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March 21, 2013

Road map

- **The algebra of the simplex method**.
- The tableau approach.
- The second example.

The implementation of the simplex method

- The idea is intuitive and simple, but how to **implement** it?
- Now we need mathematics, in particular, **linear algebra**.
- Consider a standard form linear program

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

- We may assume that all **rows** of A are **linearly independent**.
- Given a basis B and the set of nonbasic variables N, how may we determine the **entering** and **leaving** variables?
  - At this moment, treat B as given. We will discuss how to find an initial basis later.

Splitting into basic and nonbasic sets

- First, given the basis B, we may split x into (x_B, x_N), where x_B includes basic variables and x_N includes nonbasic variables.
- We may also split c into (c_B, c_N) and A into (A_B, A_N).
  - $c_B \in \mathbb{R}^{1\times m}$, $c_N \in \mathbb{R}^{1\times(n-m)}$, $A_B \in \mathbb{R}^{m\times m}$, and $c_N \in \mathbb{R}^{(n-m)\times m}$.
- As an example, consider

$$
\begin{array}{rrcrcrcrcrcl}
\min & -x_1 & & & & & & & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\
 & 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\
 & & & x_2 & & & & & + & x_5 & = & 3 \\
 & & x_i \geq 0 & \multicolumn{9}{l}{\forall i = 1, ..., 5.}
\end{array}
$$

Splitting into basic and nonbasic sets

- In the matrix representation, we have

$$c = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

- If $x_B = (x_1, x_4, x_5)$ and $x_N = (x_2, x_3)$ we have

$$c_B = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} 0 & 0 \end{bmatrix},$$
$$A_B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

  - **Orders** of variables in x_B and x_N affect these parameter matrices!

Reducing the formulation

- With the split, the linear program becomes

$$\begin{aligned} \min \quad & c_B x_B + c_N x_N \\ \text{s.t.} \quad & A_B x_B + A_N x_N = b \\ & x_B, x_N \geq 0. \end{aligned}$$

- For constraints, we may obtain $x_B = A_B^{-1}(b - A_N x_N)$. We may then plug in this into the objective function and get

$$\begin{aligned} \min \quad & c_B \big[A_B^{-1}(b - A_N x_N) \big] + c_N x_N \\ \text{s.t.} \quad & x_B = A_B^{-1}(b - A_N x_N) \\ & x_B, x_N \geq 0. \end{aligned}$$

  - A_B is indeed a square matrix. But why A_B is invertible?

Reducing the formulation

- With some more algebra, the linear program becomes

$$\begin{aligned} \min \quad & c_B A_B^{-1} b - \left(c_B A_B^{-1} A_N - c_N\right) x_N \\ \text{s.t.} \quad & x_B = A_B^{-1} b - A_B^{-1} A_N x_N \\ & x_B, x_N \geq 0. \end{aligned}$$

- Note that $x_N = 0$ (a zero vector), so for the current basis B:
  - The values of the basic variables are $x_B = A_B^{-1} b$.
  - The objective value is $z = c_B A_B^{-1} b$.
- We will use z to denote the objective value for **a given basis** and z^* to denote the objective value for the **optimal basis**.

Utilizing the new representation

- As an example, consider

$$\begin{array}{rrcrcrcrcrcl} \min & -x_1 \\ \text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\ & 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\ & & & x_2 & & & & & + & x_5 & = & 3 \\ & & x_i \geq 0 & \forall i = 1, ..., 5. \end{array}$$

- For $x_B = (x_1, x_4, x_5)$ and $x_N = (x_2, x_3)$, we have

$$A_B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix},$$

$$c_B = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad c_N = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Utilizing the new representation

- It then follows that

$$x_B = A_B^{-1} b = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_4 \\ x_5 \end{bmatrix}$$

  and

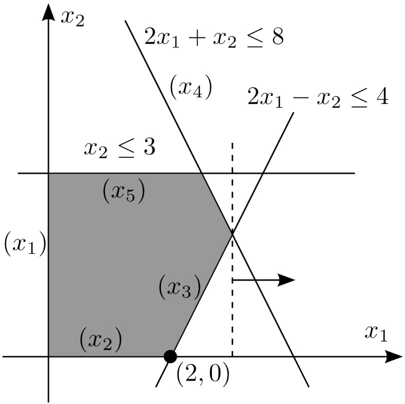
$$z = c_B A_B^{-1} b = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = -2.$$

- The current basic feasible solution is

$$x = (x_1, x_2, x_3, x_4, x_5) = (2, 0, 0, 4, 3).$$

Utilizing the new representation

- At the point (2, 0):
  - It corresponds to the bfs $x = (2, 0, 0, 4, 3)$.
  - Indeed those two binding constraints correspond to variables x_2 and x_3.
  - That is, $B = \{x_1, x_4, x_5\}$.
  - $x_4 > 0$ and $x_5 > 0$: There are positive "distances" between (2, 0) and the two corresponding constraints.



Reduced costs

- Look at the coefficient of x_N in the objective function again:

$$\min \quad c_B A_B^{-1} b - \left(c_B A_B^{-1} A_N - c_N\right) x_N.$$

  We define the **reduced cost**: $\bar{c}_N \equiv c_B A_B^{-1} A_N - c_N$.
- $\bar{c}_N \in \mathbb{R}^{1 \times (n-m)}$ is a **vector**:

$$\underbrace{\begin{bmatrix} \quad c_B \quad \end{bmatrix}}_{1 \times m} \underbrace{\begin{bmatrix} \\ \quad A_B^{-1} \quad \\ \\ \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} \\ \quad\quad A_N \quad\quad \\ \\ \end{bmatrix}}_{m \times (n-m)} - \underbrace{\begin{bmatrix} \quad\quad c_N \quad\quad \end{bmatrix}}_{1 \times (n-m)}.$$

- Each element of $\bar{c}_N$ is a coefficient of one nonbasic variable. For **one** nonbasic variable $x_j \in N$, its coefficient is

$$\bar{c}_j = \begin{bmatrix} \quad c_B \quad \end{bmatrix} \begin{bmatrix} \\ \quad A_B^{-1} \quad \\ \\ \end{bmatrix} \begin{bmatrix} \\ A_j \\ \\ \end{bmatrix} - \begin{bmatrix} c_j \end{bmatrix}.$$

Reduced costs

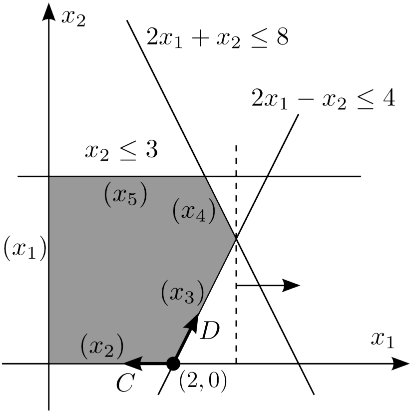
- For the same example with $N = \{x_2, x_3\}$, note that

$$\begin{aligned}\bar{c}_N &= c_B A_B^{-1} A_N - c_N \\ &= \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.\end{aligned}$$

  - So the reduced cost of x_2 is $\bar{c}_2 = \frac{1}{2}$ and that of x_3 is $\bar{c}_3 = -\frac{1}{2}$.
  - This is the amount of **reduction in cost** by increasing x_j by 1.
- We will choose x_2 as the entering variable. Why?
- In general there may be **multiple** nonbasic variables having positive reduced costs. In that case, we need a **selection rule**.

Reduced costs

- At the point (2, 0):
  - Entering x_2 means moving along direction D, which is indeed improving ($\bar{c}_2 > 0$).
  - Entering x_3 means moving along direction C, which makes things worse ($\bar{c}_3 < 0$).
- Now we know how to find an entering variable **algebraically**.



The leaving variable

- Suppose $\bar{c}_j > 0$ and we have decided to let x_j enter.
- How to choose the **leaving variable**?
- Let's look at the constraints. Because $x_B \geq 0$, we have

$$x_B = A_B^{-1}b - A_B^{-1}A_N x_N \geq 0.$$

- Increasing x_j is certainly good (because $\bar{c}_j > 0$), but that may **violate the constraints** (i.e., make a basic variable negative).
- We want the largest improvement, so we should keep increasing x_j **until** a basic variable becomes **zero**.
  - That basic variable will **leave** the basis.
  - How to find that basic variable?

The leaving variable

- $A_B^{-1}b \in \mathbb{R}^{m\times 1}$, $A_B^{-1}A_N \in \mathbb{R}^{m\times(n-m)}$, and $x_N \in \mathbb{R}^{(n-m)\times 1}$.
- When x_j increases and all other nonbasic variables remain 0:

$$\underset{m\times 1}{\begin{bmatrix} A_B^{-1}b \end{bmatrix}} - \underset{m\times(n-m)}{\begin{bmatrix} \quad A_B^{-1}A_N \quad \end{bmatrix}} \underset{(n-m)\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ x_j \\ 0 \end{bmatrix}} = \underset{m\times 1}{\begin{bmatrix} A_B^{-1}b \end{bmatrix}} - \underset{m\times 1}{\begin{bmatrix} A_B^{-1}A_j \end{bmatrix}} x_j.$$

  - A_j is the jth column of matrix A.
- To determine which basic variable will become 0 first, we may do a **ratio test**.

The ratio test

- While increasing x_j, we need to make sure that

$$\begin{bmatrix} \\ A_B^{-1}b \\ \\ \end{bmatrix} - \begin{bmatrix} \\ A_B^{-1}A_j \\ \\ \end{bmatrix} x_j \geq 0.$$

  - Note that $A_B^{-1}b \geq 0$ but $A_B^{-1}A_j$ is not.
  - For element i, if $(A_B^{-1}A_j)_i \leq 0$, then $A_B^{-1}b - A_B^{-1}A_j x_j$ will **never be negative** when we increase x_j.
- For those k such that $(A_B^{-1}A_j)_k > 0$, we define

$$\theta_k \equiv \frac{(A_B^{-1}b)_k}{(A_B^{-1}A_j)_k} \quad \forall k : (A_B^{-1}A_j)_k > 0.$$

- Then the ith row will become 0 first if and only if

$$\theta_i \leq \theta_k \quad \forall k : (A_B^{-1}A_j)_k > 0.$$

The ratio test

- For the same example with $N = \{x_2, x_3\}$, we have decided to let x_2 enters. Then we have

$$A_B^{-1}A_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix} \text{ and } A_B^{-1}b = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}.$$

  Note that the three rows are for x_1, x_4, and x_5, respectively.

- So the relevant ratios are

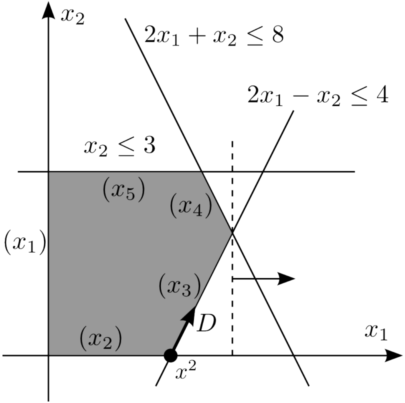
$$\theta_2 = \frac{4}{2} = 2 \quad \text{and} \quad \theta_3 = \frac{3}{1} = 3.$$

  The ratio for x_1 is irrelevant because $-\frac{1}{2} \leq 0$.

- As $\theta_2 = 2 < 3 = \theta_3$, x_4 (the basic variable associated with the second row) will be the leaving variable.
- In general, if there is a tie, a selection rule must be specified.

The ratio test

- At the point (2, 0), along direction D:
  - $A_B^{-1}A_2 = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$ and $A_B^{-1}b = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.
  - We will hit $2x_1 + x_2 \leq 8$ before $2x_1 - x_2 \leq 4$.
  - We know this as $\theta_2 < \theta_3$.
  - The last nonbinding constraint, $x_1 \geq 0$, will never be hit.
  - We know this as $-\frac{1}{2} \leq 0$.



Summary of the simplex method

- For a minimization LP with an optimal solution (i.e., neither infeasible nor unbounded) and an initial basis B:
- Start from B and the corresponding set of nonbasic variables N.
- Repeat:
  - Calculate the reduced costs
  $$\bar{c}_N = c_B A_B^{-1} A_N - c_N.$$
  Choose an entering variable x_j that has $\bar{c}_j > 0$.
  - If $\bar{c}_N \leq 0$, stop and report the current bfs as optimal.
  - Do the ratio test by calculating
  $$\theta_k \equiv \frac{(A_B^{-1} b)_k}{(A_B^{-1} A_j)_k} \quad \forall k : (A_B^{-1} A_j)_k > 0.$$
  Choose a leaving variable x_i who has the minimum relevant ratio.
  - Switch x_i and x_j in B and N.

Summary of the simplex method

- At each iteration:
  - First check what are B and N.
  - **Read** A_B, A_N, c_B, c_N, and b from the original formulation.
  - **Calculate** $z = c_B A_B^{-1} A_N$, $\bar{c}_N = c_B A_B^{-1} A_N - c_N$, $x_B = A_B^{-1} b$, and $A_B^{-1} A_N$.
  - These four things will **change** whenever the basis changes
- For maximization problems:
  - Change it to a minimization problem.
  - Choose a **negative** reduced cost for the entering variable.

Summary of the simplex method

- The idea of the simplex method is simple:
  - Move along **edges**.
  - Search for improving directions **greedily**.
  - Stop when no way to improve.
- Implementing the simplex method requires linear algebra.
  - But the arithmetic requires only **inverse** and **matrix multiplication**.

Summary of the simplex method

- Some things are still missing:
  - How to obtain the initial basis?
  - Why A_B is invertible?
  - What if there are multiple choices for entering and leaving variables?
  - May we know whether the optimal solution is unique?
  - What if the linear program is infeasible or unbounded?
- We will answer some of these questions later. Before that, let's get more familiar with the simplex method by studying the **tableau approach**.

Road map

- The algebra of the simplex method.
- **The tableau approach**.
- The second example.

Reduced standard form

- Recall that a standard form LP $\min\{cx | Ax = b, x \geq 0\}$ can be expressed as

$$\begin{array}{rl} \min & c_B A_B^{-1} b - \left(c_B A_B^{-1} A_N - c_N\right) x_N \\ \text{s.t.} & x_B = A_B^{-1} b - A_B^{-1} A_N x_N \\ & x_B, x_N \geq 0. \end{array}$$

- We may further reduce it to

$$\begin{array}{rcccccc} \min & & - & \left(c_B A_B^{-1} A_N - c_N\right) x_N & + & c_B A_B^{-1} b \\ \text{s.t.} & I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b \\ & x_B, x_N \geq 0. & & & & \end{array}$$

Tableau

- The last form can be organized into a **tableau**:

(1)	0	$c_B A_B^{-1} A_N - c_N$	$c_B A_B^{-1} b$
(m)	I	$A_B^{-1} A_N$	$A_B^{-1} b$
	(m)	$(n-m)$	(1)

- Changing B and N requires to update $c_B A_B^{-1} A_N - c_N$, $c_B A_B^{-1} b$, $A_B^{-1} A_N$, and $A_B^{-1} b$. Now these can be done by doing **elementary row operations** on the tableau.

Tableau

- Consider the linear program

$$\begin{aligned} \min \quad & -2x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\ & 2x_1 + x_2 \leq 8 \\ & x_i \geq 0 \quad \forall\ i = 1, 2. \end{aligned}$$

  and its standard form

$$\begin{array}{lrcrcrcrcr} \min & -2x_1 & - & 3x_2 & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & = & 6 \\ & 2x_1 & + & x_2 & & & + & x_4 & = & 8 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall\ i = 1, ..., 4.} \end{array}$$

Initial bases

- We need an initial basis that gives us a basic feasible solution.
- $B = \{x_3, x_4\}$ must give us a basic feasible solution. Why?
- For any original LP with **only no-greater-than** constraints and **nonnegative RHS**, we may select all slack variables to form our initial basis.
- The initial tableau is

2	3	0	0	0
1	2	1	0	$x_3 = 6$
2	1	0	1	$x_4 = 8$

  - The basic columns have zeros in the 0th row and an identity matrix in the other rows.
  - The identity matrix associates each row with a basic variable.
  - Numbers in the 0th row for nonbasic columns are reduced costs.

Iterations: the entering variable

- How to find an entering variable in a tableau?
- All we need to do is to find a **positive** value in the 0th row!
  - In the 0th row, nonbasic columns contain **reduced costs**.
  - To decrease the objective value, we need a positive reduced cost.
- In this example, we may enter either x_1 or x_2.
- We have not introduced any selection rule. Let's just choose x_1.

2	3	0	0	0
1	2	1	0	$x_3 = 6$
2	1	0	1	$x_4 = 8$

Iterations: the leaving variable

- How to find a leaving variable in a tableau?
- All we need to do is a **ratio test**:
  - Divide the RHS column by the entering column.
  - Among those rows with a **positive denominator** (the value in the entering column), we find one that has **the smallest ratio**.
- In this example, x_4 leaves because $\frac{8}{2} < \frac{6}{1}$.

2	3	0	0	0
1	2	1	0	$x_3 = 6$
$\boxed{2}$	1	0	1	$x_4 = 8$

Iterations: pivoting

- Once we determine the entering and leaving variables, we find the **pivot**.
  - The intersection of the entering column and the leaving row.
- To move to the next basic feasible solution, we need to **make the entering column a basic column**:
  - The pivot should become 1.
  - All other numbers in that row should become 0.
  - Do this through elementary row operations.
- In this example:

<table>
<tr><td>

2	3	0	0	0
1	2	1	0	$x_3 = 6$
$\boxed{2}$	1	0	1	$x_4 = 8$

</td><td>$\rightarrow$</td><td>

0	2	0	-1	-8
0	$\frac{3}{2}$	1	$-\frac{1}{2}$	$x_3 = 2$
1	$\frac{1}{2}$	0	$\frac{1}{2}$	$x_1 = 4$

</td></tr>
</table>

Iterations

- Let's do one more iteration.
  - Entering variable?
  - Leaving variable?
- In this example:

$$\begin{array}{cccc|c} 0 & 2 & 0 & -1 & -8 \\ \hline 0 & \boxed{\frac{3}{2}} & 1 & -\frac{1}{2} & x_3 = 2 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & x_1 = 4 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} 0 & 0 & -\frac{4}{3} & -\frac{1}{3} & -\frac{32}{3} \\ \hline 0 & 1 & \frac{2}{3} & -\frac{1}{3} & x_2 = \frac{4}{3} \\ 1 & 0 & -\frac{1}{3} & \frac{2}{3} & x_1 = \frac{10}{3} \end{array}$$

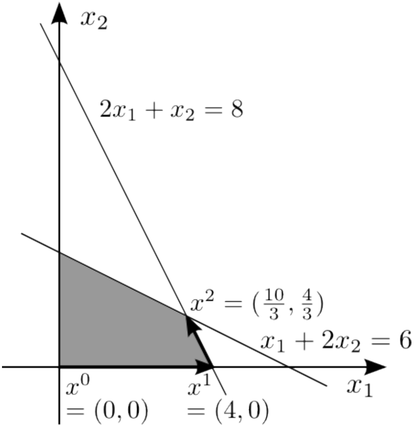
- Stop or keep iterating?
- We have found the optimal tableau, which implies that the optimal basic feasible solution is $x^* = (\frac{10}{3}, \frac{4}{3}, 0, 0)$.
- The objective value is $-2 \times \frac{10}{3} - 3 \times \frac{4}{3} = -\frac{32}{3}$. Coincident?

Summary

- To use the tableau approach for a minimization problem (with a given basis) which has an optimal solution:
  1. Find the standard form.
  2. Copy numbers into the tableau but negate the objective coefficients.
  3. Repeat:
     3.1 Find a positive number in the 0th row for an entering variable. If there is none, stop and report the optimal solution.
     3.2 Do a ratio test for a leaving variable.
     3.3 Pivoting: Make the entering column a basic column.
- How about maximization problems?
  - Just replace "positive" by "negative" in Step 3.1.

Visualizing the iterations

- Let's visualize this example and relate basic feasible solutions with extreme points:
- The initial tableau corresponds to the origin $x^0 = (0, 0)$.
- After one iteration, we move to $x^1 = (4, 0)$.
- After two iterations, we move to $x^2 = (\frac{10}{3}, \frac{4}{3})$, which is optimal.



Road map

- The algebra of the simplex method.
- The tableau approach.
- **The second example**.

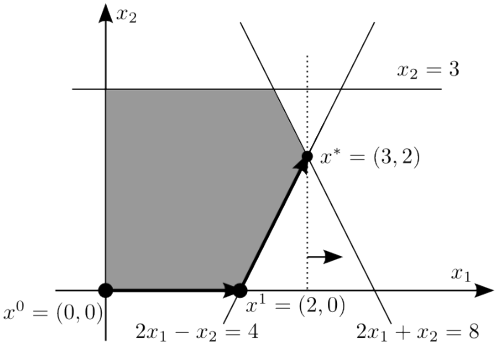
The second example

- Consider another example:

$$
(P) \quad \begin{array}{rrcrcrl}
\max & x_1 & & & & & \\
\text{s.t.} & 2x_1 & - & x_2 & \leq & 4 & \text{(Constraint 1)} \\
 & 2x_1 & + & x_2 & \leq & 8 & \text{(Constraint 2)} \\
 & & & x_2 & \leq & 3 & \text{(Constraint 3)} \\
 & & & x_i & \geq & 0 & \forall\, i = 1, 2.
\end{array}
$$

Initialization

- Looking at the graphical solution for (P), we may see that its optimal solution is $x^* = (3, 2)$. The dotted line is the isoprofit line. The short arrow indicates the direction we push the isoprofit line.



Initialization

- The standard form of problem (P) is

$$
(S) \quad \begin{array}{rrcrcrcrcrcl}
\max & x_1 \\
\text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\
& 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\
& & & x_2 & & & & & + & x_5 & = & 3 \\
& \multicolumn{11}{l}{x_i \geq 0 \quad \forall\, i = 1, ..., 5.}
\end{array}
$$

The first iteration

- For problem (S), we form the initial tableau

$$
\begin{array}{ccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
\hline
2 & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
$$

  - The initial basic feasible solution (bfs) is $x^0 = (0, 0, 4, 8, 3)$.
  - The current objective value $z_0 = 0$.
  - Basic variables are x_3, x_4, and x_5.
  - Nonbasic variables are x_1 and x_2.
- In the graph of (P), we may see that x^0 is the origin.

The first iteration

- The entering variable is x_1 because it is the only nonbasic variable that has a negative reduced cost. Note that this is a maximization problem!
- The leaving variable is x_3 according to the ratio test. Note that row 3 does not participate in the ratio test. Why?
- The next tableau is found by pivoting at 2:

$$
\begin{array}{ccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 \\
\hline
\boxed{2} & -1 & 1 & 0 & 0 & x_3 = 4 \\
2 & 1 & 0 & 1 & 0 & x_4 = 8 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 2 \\
\hline
1 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\
0 & 2 & -1 & 1 & 0 & x_4 = 4 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
$$

- The current bfs becomes $x^1 = (2, 0, 0, 4, 3)$ and the current objective value becomes $z_1 = 2$.

The second iteration

- The entering variable is x_2 because its reduced cost is negative.
- The leaving variable is x_4 according to the ratio test. Note that row 1 does not participate in the ratio test. Why?
- The second iteration is

$$
\begin{array}{ccccc|c}
0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 2 \\
\hline
1 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & x_1 = 2 \\
0 & \boxed{2} & -1 & 1 & 0 & x_4 = 4 \\
0 & 1 & 0 & 0 & 1 & x_5 = 3
\end{array}
\quad \rightarrow \quad
\begin{array}{ccccc|c}
0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 3 \\
\hline
1 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & x_1 = 3 \\
0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 & x_2 = 2 \\
0 & 0 & \frac{1}{2} & \frac{-1}{2} & 1 & x_5 = 1
\end{array}
$$

  and we get the third bfs $x^* = (3, 2, 0, 0, 1)$, which is optimal, and the optimal objective value $z^* = 3$.
- As no nonbasic variable has a negative reduced cost, we conclude that the current basis is optimal.

Verifying our solution

- The three basic feasible solutions we obtain are
  - $x^0 = (0, 0, 4, 8, 3)$.
  - $x^1 = (2, 0, 0, 4, 3)$.
  - $x^* = (3, 2, 0, 0, 1)$.

  Do they fit our graphical approach?